

# Exit problems associated with affine reflection groups

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## Abstract

We obtain a formula for the distribution of the first exit time of Brownian motion from the alcove of an affine Weyl group. In most cases the formula is expressed compactly, in terms of Pfaffians. Expected exit times are derived in the type  $A$  case. The results extend to other Markov processes. We also give formulas for the real eigenfunctions of the Dirichlet and Neumann Laplacians on alcoves, observing that the ‘Hot Spots’ conjecture of J. Rauch is true for alcoves.

## 1 Introduction

The distribution of the first exit time of Brownian motion from the interval  $(0,1)$  may be obtained by the reflection principle. If  $B$  is a Brownian motion,  $T_i$  the hitting time of the level  $i$ ,  $T_{0,1} := T_0 \wedge T_1$ , and  $\mathbb{P}_x$  denotes the law of  $B$  started at  $x \in (0,1)$ , then

$$\mathbb{P}_x(T_{0,1} > t) = \sum_{n \in \mathbb{Z}} [\mathbb{P}_x(B_t \in 2n + (0,1)) - \mathbb{P}_x(B_t \in 2n - (0,1))]. \quad (1)$$

Using cancellation and the reflection principle, formula (1) may be rewritten as  $\mathbb{P}_x(T_{0,1} > t) = \phi(x, t)$ , where

$$\phi(x, t) = \mathbb{P}_x(T_0 > t) + \sum_{n=1}^{\infty} (-1)^n [\mathbb{P}_x(T_{-i} > t) - \mathbb{P}_x(T_i > t)]. \quad (2)$$

More generally, suppose that  $B$  is a standard Brownian motion in a real Euclidean space  $V$ . If  $B$  is started inside the alcove  $\mathcal{A}$  of an affine reflection group acting on  $V$ , there exists an expression analogous to (1) for the distribution of the first exit time of  $B$  from  $\mathcal{A}$ , which is given later in equation (14). The expression involves integration of the Gaussian kernel over the multi-dimensional alcove. The aim of this paper is to give a formula analogous to (2)—that is, a formula for the exit probability of standard Brownian motion from the alcove, in terms of exit probabilities from simpler domains.

As an example, in the type  $\tilde{A}$  case this formula involves only one-dimensional exit probabilities and can be written in terms of Pfaffians (see appendix for the definition of a Pfaffian).

To put our results in context, we state the following proposition. Let  $B_1, \dots, B_k$  be independent standard Brownian motions started at  $x_1, \dots, x_k \in \mathbb{R}$  and let  $(\xi_n = e^{i2\pi B_n})_{n \in [k]}$  be their projections onto the circle, where  $[k] := \{1, \dots, k\}$ . Define the times of first collision

$$\begin{aligned} T_{ij} &= \inf\{t : B_i(t) = B_j(t)\} & \tilde{T}_{ij} &= \inf\{t : \xi_i(t) = \xi_j(t)\} \\ T &= \min\{T_{ij} : 1 \leq i < j \leq k\} & \tilde{T} &= \min\{\tilde{T}_{ij} : 1 \leq i < j \leq k\}. \end{aligned}$$

Then  $T$  is equal to the first exit time of  $k$  dimensional Brownian motion started at  $x = (x_1, \dots, x_k)$  from a chamber of type  $A_{k-1}$ , and it was proved in [8] that

$$\mathbb{P}_x(T > t) = \begin{cases} \text{Pf}(p_{ij})_{i,j \in [k]} & (k \text{ even}) \\ \sum_{l=1}^k (-1)^{l+1} \text{Pf}(p_{ij})_{i,j \in [k] \setminus \{l\}} & (k \text{ odd}) \end{cases} \quad (3)$$

where for  $i < j$ ,  $p_{ij} = \mathbb{P}_x(T_{ij} > t)$  and  $p_{ji} = -p_{ij}$ . As observed in [11],  $\tilde{T}$  is equal to the first exit time of the Brownian motion from an alcove of type  $\tilde{A}_{k-1}$ . A special case of our main result gives a companion to (3):

**Proposition 1.** (i) *If  $k$  is even then*

$$\mathbb{P}_x(\tilde{T} > t) = \text{Pf}(\tilde{p}_{ij})_{i,j \in [k]}$$

where for  $i < j$ ,  $\tilde{p}_{ij} = \mathbb{P}_x(\tilde{T}_{ij} > t)$  and  $\tilde{p}_{ji} = -\tilde{p}_{ij}$ .

(ii) *If  $k$  is odd then*

$$\mathbb{P}_x(\tilde{T} > t) = \sum_{l=1}^k (-1)^{l+1} \text{Pf}(q_{ij})_{i,j \in [k] \setminus \{l\}}$$

where for  $i < j$ ,  $q_{ij} = \mathbb{P}_x(\tilde{T}_{ij} > t) + 2\mathbb{P}_x(\tilde{T}_{ij} \leq t, \tilde{T}_{ij} < T_{ij})$  and  $q_{ji} = -q_{ij}$ .

The relationship between  $q_{ij}$  in Proposition 1 and  $p_{ij}$  in (3) is clarified by noting that  $B_i - B_j$  is also a Brownian motion and

$$\begin{aligned} \psi(x, t) &:= \mathbb{P}_x(T_{0,1} > t) + 2\mathbb{P}_x(T_{0,1} \leq t, T_1 < T_0) \\ &= \mathbb{P}_x(T_0 > t) + \sum_{n=1}^{\infty} [\mathbb{P}_x(T_{-i} > t) - \mathbb{P}_x(T_i > t)], \end{aligned} \quad (4)$$

which is proved in Lemma 25 and may be compared with (2). In the case  $k = 3$ ,  $\tilde{T}$  equals the first exit time of Brownian motion from an equilateral triangle, which is the alcove of type  $\tilde{A}_2$ . This relates to scaling limits occurring in, for example, a three player gambler's ruin problem and a three

tower problem [1, 5]. As a further example, if the alcove is of type  $\tilde{C}_k$  then  $\tilde{T}$  relates to first collision times for  $k$  independent standard Brownian motions on the interval.

The expected exit time is obtained in the type  $\tilde{A}$  case, and also a generalisation of de Bruijn's formula for multiple integrals involving determinants. The present work extends that in [8], where the authors consider the exit time from a chamber - that is, an unbounded domain which is the fundamental region of a finite reflection group. The extension to  $\tilde{A}_{k-1}$  with odd  $k$  was prompted by Neil O'Connell, who suggested the solution for  $k = 3$ .

The rest of the paper is organised as follows. Sections 2 and 3 present necessary background material, the main results with applications, a general reflection principle and an affine generalisation of De Bruijn's formula. Details of the main result in the different type cases are given in section 4. Proofs are contained in section 5, and the real eigenfunctions of the Laplacian with Dirichlet and Neumann boundary conditions are considered in section 6.

## 2 The geometric setting

### 2.1 Finite Weyl groups and chambers

Background on root systems and reflection groups may be found in, for example, [12]. Let  $V$  be a real Euclidean space with a positive symmetric bilinear form  $\langle x, y \rangle$ . Let  $\Phi$  be an irreducible crystallographic root system in  $V$  with associated reflection group  $W$ . Let  $\Delta$  be a simple system in  $\Phi$  with corresponding positive system  $\Phi^+$  and fundamental chamber

$$\mathcal{C} = \{x \in V : \forall \alpha \in \Delta, \langle \alpha, x \rangle > 0\}.$$

We will call  $\Phi^\vee$  the set of coroots  $\alpha^\vee = 2\alpha/\langle \alpha, \alpha \rangle$  for  $\alpha \in \Phi$ . Then,  $L := \mathbb{Z}$ -span of  $\Phi^\vee$  is a  $W$ -stable lattice called the coroot lattice. For  $\alpha \in \Phi$  and  $x \in V$  we make the definitions

$$\begin{aligned} H_\alpha &= \{y \in V : \langle \alpha, y \rangle = 0\} \\ s_\alpha(x) &= x - \langle \alpha, x \rangle \alpha^\vee. \end{aligned}$$

Thus  $s_\alpha$ ,  $\alpha \in \Phi^+$  are the reflections in  $W$ .

### 2.2 Affine Weyl groups and alcoves

The affine Weyl group  $W_a$  associated with  $\Phi$  is the group generated by all affine reflections with respect to the hyperplanes  $H_{(\alpha, n)} = \{x \in V : \langle x, \alpha \rangle = n\}$ ,  $\alpha \in \Phi^+$ ,  $n \in \mathbb{Z}$ . It has a semi-direct product decomposition in terms of the Weyl group  $W$  and the coroot lattice  $L$ : each element of  $W_a$  may be written uniquely as  $\tau(l)w$ , where  $w \in W$  and  $\tau(l)$  is the translation by

$l \in L$ . We may therefore attribute a sign to each  $w_a = \tau(l)w \in W_a$  by  $\varepsilon(w_a) = \varepsilon(w) := \det(w)$ . The fundamental alcove is the bounded domain defined by

$$\begin{aligned}\mathcal{A} &= \{x \in V : \forall \alpha \in \Phi^+, 0 < \langle x, \alpha \rangle < 1\} \\ &= \{x \in V : \langle x, \tilde{\alpha} \rangle < 1 \text{ and } \forall \alpha \in \Delta, \langle x, \alpha \rangle > 0\}\end{aligned}$$

where  $\tilde{\alpha}$  is the highest positive root.

## 2.3 Affine root systems

We refer to [13] for this formalism although we use slightly modified notations for the sake of consistency.

**Definition 2.** *If  $\Phi$  is an irreducible crystallographic root system as previously introduced, the corresponding affine root system is  $\Phi_a := \Phi \times \mathbb{Z}$ . For  $\lambda = (\alpha, n) \in \Phi_a$  and  $x \in V$  we define*

$$\begin{aligned}\lambda(x) = \lambda.x &= \langle \alpha, x \rangle - n \\ H_\lambda &= \{y \in V : \lambda.y = 0\} \\ s_\lambda(x) &= x - (\lambda.x)\alpha^\vee\end{aligned}$$

Thus  $s_\lambda$  is the reflection with respect to the hyperplane  $H_\lambda$ , and we may write  $s_\lambda = \tau(n\alpha^\vee)s_\alpha$ . Writing  $w_a = \tau(l)w \in W_a$ , we have that  $W_a$  acts on  $V$  by  $w_a(x) = w(x) + l$ ; we define further the action of  $W_a$  on  $\Phi_a$  by

**Definition 3.** *For  $w_a = \tau(l)w \in W_a$  and  $\lambda = (\alpha, n) \in \Phi_a$ ,*

$$w_a(\lambda) = (w\alpha, n + \langle w\alpha, l \rangle) \in \Phi_a.$$

We then have  $w_a(\lambda).x = \lambda.w_a^{-1}(x)$  for  $w_a \in W_a$ ,  $\lambda \in \Phi_a$ ,  $x \in V$ , which is analogous to the fact that  $W$  is a group of isometries; we also have  $w_a H_\lambda = H_{w_a(\lambda)}$ . If  $\lambda = (\alpha, m)$ ,  $\mu = (\beta, n) \in \Phi_a$  then we will refer to the angle between  $\lambda$  and  $\mu$ , meaning the angle between  $\alpha$  and  $\beta$ ; by  $\lambda \perp \mu$  we mean  $\langle \alpha, \beta \rangle = 0$ . The usual properties of a reflection are then preserved:  $s_\lambda(\lambda) = (-\alpha, -n) =: -\lambda$  and  $s_\lambda(\mu) = \mu$  if  $\lambda \perp \mu$ .

**Definition 4.** *The affine simple system is  $\Delta_a := \{(\alpha, 0), \alpha \in \Delta; (-\tilde{\alpha}, -1)\}$  and the corresponding positive system is  $\Phi_a^+ := \{(\alpha, n) : (n = 0 \text{ and } \alpha \in \Phi^+) \text{ or } n \leq -1\}$ .*

This definition is tailor-made so that

$$\mathcal{A} = \{x \in V : \forall \lambda \in \Phi_a^+, \lambda(x) > 0\} = \{x \in V : \forall \lambda \in \Delta_a, \lambda(x) > 0\}.$$

### 3 Background and main results

We present here our main results, which extend the main result in [8] to the affine cases. In section 3.5 we give some applications in the type  $\tilde{A}$  case.

#### 3.1 Consistency

Let  $(\mathcal{W}, \phi, \phi^+, \delta, F) \in \{(W, \Phi, \Phi^+, \Delta, \mathcal{C}), (W_a, \Phi_a, \Phi_a^+, \Delta_a, \mathcal{A})\}$  and for  $I \subset \phi^+$  define  $\mathcal{W}^I = \{w \in \mathcal{W} : wI \subset \phi^+\}$  and  $\mathcal{I} = \{wI : w \in \mathcal{W}^I\}$ . For  $S \subset \phi$ , we define the set of orthogonal subsets of  $S$ :

$$\mathcal{O}(S) := \{Y \subset S : \forall \lambda \neq \mu \in Y, \lambda \perp \mu\}.$$

**Definition 5 (Consistency).** • We will say that  $I$  satisfies hypothesis (C1) if there exists  $J \in \mathcal{O}(\delta \cap I)$  such that if  $J \subset A \in \mathcal{I}$  then  $A = I$ .

- We will say that  $I$  satisfies hypothesis (C2) if the restriction of the determinant to the subgroup  $U = \{w \in \mathcal{W} : wI = I\}$  is trivial, i.e.  $\forall w \in U, \varepsilon(w) = \det w = 1$ .
- We will say that  $I$  satisfies hypothesis (C3) if  $\mathcal{I}$  is finite.
- $I$  will be called **consistent** if it satisfies (C1), (C2) and (C3).

Condition (C2) makes it possible to attribute a sign to every element of  $\mathcal{I}$  by  $\varepsilon_A := \varepsilon(w)$  for  $A \in \mathcal{I}$ , where  $w$  is any element of  $\mathcal{W}^I$  with  $wI = A$ .

#### 3.2 Reflectability

Let  $X = (X_t, t \geq 0)$  be a  $V$ -valued process and let  $\mathbb{P}_x$  denote the law of  $X$  started at  $x \in F$ . We will call  $X$  **reflectable** if it satisfies the conditions of the following:

**Definition 6 (Reflectable process).** •  $X$  has the strong Markov property.

- The sample paths of  $X$  are almost surely continuous.
- The law of  $X$  is  $\mathcal{W}$ -invariant - that is,  $\mathbb{P}_x \circ (wX)^{-1} = \mathbb{P}_{wx} \circ X^{-1}$  for all  $w \in \mathcal{W}, x \in V$ .

#### 3.3 Exit times

We now introduce some notation for exit times. Let  $X$  be a reflectable process in  $V$ . For convenience we may write each  $\lambda \in \phi^+$  in the form  $(\alpha, n)$  by identifying  $\alpha \in \Phi$  with  $(\alpha, 0) \in \Phi_a$ . Then for  $\lambda = (\alpha, n) \in \phi^+$  define  $T_\lambda = \inf\{t \geq 0 : \lambda.X_t = 0\}$  and for  $A = \{\lambda_1, \dots, \lambda_k\} \subset \phi^+$  write  $T_A := T_{\lambda_1, \dots, \lambda_k} := \min_{\lambda \in A} T_\lambda$ . Finally, let  $T$  denote the first exit time

of  $X$  from the fundamental chamber  $\mathcal{C}$ —that is,  $T = T_\delta$  in the finite case  $(\mathcal{W}, \phi^+, \delta, F) = (W, \Phi^+, \Delta, \mathcal{C})$ ; and let  $\tilde{T}$  denote the first exit time of  $X$  from the fundamental alcove  $\mathcal{A}$ —that is,  $\tilde{T} = T_\delta$  in the affine case  $(\mathcal{W}, \phi^+, \delta, F) = (W_a, \Phi_a^+, \Delta_a, \mathcal{A})$ .

### 3.4 Main results

The following Theorem extends the main result of [8] to include those affine Weyl groups which have a consistent subset; the details of its application to particular affine Weyl groups are given in section 4. Theorem 8 deals with an important case where a consistent subset is not available.

**Theorem 7.** *Suppose  $I$  is consistent,  $X$  is reflectable and  $x \in F$ . Then :*

$$\mathbb{P}_x(T_\delta > t) = \sum_{A \in \mathcal{I}} \varepsilon_A \mathbb{P}_x(T_A > t). \quad (5)$$

Note that the sum is finite even for affine Weyl groups. In the  $\tilde{A}_{k-1}$  case with odd  $k$ , no consistent subset is available and we require a different formalism: for  $A \in \mathcal{O}(\Phi)$ , define

$$\begin{aligned} E_A &= \{v \in \text{Span}(A) : \forall \beta \in A, (v, \beta) \in \mathbb{Z}\} \\ \varepsilon_v^A &= (-1)^{\#\{\beta \in A : \langle v, \beta \rangle > 0\}} \\ |v|_A &= \max\{|\langle v, \beta \rangle| : \beta \in A\} \end{aligned}$$

where  $\#$  is the cardinality function. For  $v, \beta \in V$  define

$$T_{\beta, v} = \inf\{t \geq 0 : \langle X_t, \beta \rangle = \langle v, \beta \rangle\}, \quad T_{A, v} = \min_{\beta \in A} T_{\beta, v}.$$

To clarify,  $E_A$  is a lattice (equal to the  $\mathbb{Z}$ -span of  $A/2$ ) and  $\varepsilon_v^A, |\cdot|_A$  give a sign and norm respectively on this lattice; and  $T_{A, v}$  is the first time that the projections of  $X_t$  and  $v$  coincide along some  $\beta \in A$ .

**Theorem 8.** *In the case  $\mathcal{W} = \tilde{A}_{k-1}$  with  $k$  odd, if  $X$  is reflectable and  $x \in \mathcal{A}$  then*

$$\mathbb{P}_x(\tilde{T} > t) = \sum_{A \in \mathcal{I}} \sum_{k \in \mathbb{N}} \sum_{\substack{v \in E_A \\ |v|_A = k}} \varepsilon_A \varepsilon_v^A \mathbb{P}_x(T_{A, v} > t) \quad (6)$$

*if this sum converges, where  $I$  and  $\mathcal{I}$  are taken from the case  $\mathcal{W} = A_{k-1}$ .*

#### 3.4.1 The ‘orthogonal’ case

We begin this section by recording some definitions.

**Definition 9.** • We say  $A \subset \phi^+$  is **block-orthogonal** if it can be partitioned into blocks  $(\rho_i)$  such that  $\rho_i \perp \rho_j$  for  $i \neq j$  and each  $\rho_i$  is either a singlet or a pair of roots whose mutual angle is  $\pi$ .

- We say  $A \subset \phi^+$  is **semi-orthogonal** if it can be partitioned into blocks  $(\rho_i)$  such that  $\rho_i \perp \rho_j$  for  $i \neq j$  and each  $\rho_i$  is either a singlet or a set of vectors whose mutual angles are integer multiples of  $\pi/4$ .

If  $I$  is block-orthogonal and  $X$  has independent components in orthogonal directions, (5) factorises to give

$$\mathbb{P}_x(T_\delta > t) = \sum_{A \in \mathcal{I}} \varepsilon_A \prod_i \mathbb{P}_x(T_{\rho_i} > t). \quad (7)$$

In many cases it is convenient to write (7) in terms of Pfaffians, and the details are given in section 4. Under slightly stronger conditions on  $X$ , (6) factorises analogously:

**Proposition 10.** *In the case  $\mathcal{W} = \tilde{A}_{k-1}$  with  $k$  odd, under conditions on  $X$  which hold for Brownian motion we have*

$$\mathbb{P}_x(\tilde{T} > t) = \sum_{A \in \mathcal{I}} \varepsilon_A \prod_{\beta \in A} (\mathbb{P}_x[T_\beta \wedge T_{(\beta,1)} > t] + 2\mathbb{P}_x[T_\beta > T_{(\beta,1)} \leq t]). \quad (8)$$

This expression may also be written in terms of Pfaffians, as noted in Proposition 1(ii).

### 3.5 Applications

#### 3.5.1 Expected exit time in the type $\tilde{A}$ case

The fundamental chamber for  $A_{k-1}$  is  $\mathcal{C} = \{x \in V : x_1 > x_2 > \dots > x_k\}$  where  $V = \mathbb{R}^k$  or  $V = \{x \in \mathbb{R}^k : x_1 + \dots + x_k = 0\}$ . As noted in the introduction,  $T$  is the first ‘collision time’ between any two coordinates of  $X$ . The fundamental alcove for the corresponding affine Weyl group  $\tilde{A}_{k-1}$  is  $\mathcal{A} = \{x \in V : 1 + x_k > x_1 > x_2 > \dots > x_k\}$ .

In the  $A_{k-1}$  case, an explicit formula for the expected exit time of Brownian motion from the fundamental chamber has been obtained in [8]:

$$\mathbb{E}_x(T) = \sum_{\pi \in P_2(k)} (-1)^{c(\pi)} F_p(x_\pi) \quad (9)$$

where  $p = \lfloor k/2 \rfloor$  and  $x_\pi = (x_i - x_j)_{\{i < j\} \in \pi} \in \mathbb{R}_+^p$ . Here  $P_2(k)$  is the set of partitions of  $[k] = \{1, \dots, k\}$  into  $k/2$  pairs if  $k$  is even and into  $(k-1)/2$  pairs and a singlet if  $k$  is odd. The quantity  $c(\pi)$  is the number of crossings in the partition  $\pi$  (if  $k$  is odd, we consider an extra pair made of the singlet and another singlet labelled 0, and use this pair to compute the number

of crossings); for an illustration see section 4.1. The notation  $\{i < j\} \in \pi$  means that  $\{i, j\} \in \pi$  and  $i < j$ , and the function  $F_p$  is given by

$$F_p(y_1, \dots, y_p) = \frac{2^{p+1}\Gamma(p/2)}{\pi^{p/2}(p-2)} \int_0^{y_1} \cdots \int_0^{y_p} \frac{dz_1 \dots dz_p}{(z_1^2 + \dots + z_p^2)^{p/2-1}}.$$

We prove an analogous formula:

**Proposition 11.** *In the  $\tilde{A}_{k-1}$  case, if  $X$  is Brownian motion then*

$$\mathbb{E}_x(\tilde{T}) = \sum_{\pi \in P_2(k)} (-1)^{c(\pi)} \tilde{F}_p(x_\pi)$$

where

$$\tilde{F}_p(y_1, \dots, y_p) = \frac{2^{2p}}{\pi^{p+2}} \sum_{l \in \mathbb{O}^p} \frac{1}{(l_1^2 + \dots + l_p^2)} \prod_{s=1}^p \frac{1}{l_s} \sin(\pi l_s y_s)$$

where  $\mathbb{O} = 2\mathbb{N}+1$  if  $k$  is even and  $\mathbb{O} = 2\mathbb{N}$  if  $k$  is odd, and with the definition  $\frac{1}{l_s} \sin(\pi l_s y_s) = \frac{1}{2} \pi y_s$  when  $l_s = 0$ .

In the case  $k = 3$  we will recover the known formula

$$\mathbb{E}_x(\tilde{T}) = x_{12}x_{23}(1 - x_{13}), \quad (10)$$

where  $0 < x_{ij} = x_i - x_j < 1$ , for the expected exit time of Brownian motion from an equilateral triangle.

### 3.5.2 Dual formulae and small time behaviour

Dual to (2) and (4) are the formulae

$$1 - \phi(x, t) = \mathbb{P}_x(T_0 \leq t) + \sum_{n=1}^{\infty} (-1)^n [\mathbb{P}_x(T_{-i} \leq t) - \mathbb{P}_x(T_i \leq t)] \quad (11)$$

$$1 - \psi(x, t) = \mathbb{P}_x(T_0 \leq t) + \sum_{n=1}^{\infty} [\mathbb{P}_x(T_{-i} \leq t) - \mathbb{P}_x(T_i \leq t)]. \quad (12)$$

In the block-orthogonal case of section 3.4.1, these dual formulae may be used to obtain asymptotics for the small time behaviour of the exit probability. For example, exact asymptotics can be obtained in the Brownian case, as in section 4.6.2 of [8] (we omit the details).

### 3.5.3 Eigenfunctions for alcoves

In section 6, using results from [4], we obtain formulae for the real eigenfunctions of the Laplacian on alcoves with Dirichlet or Neumann boundary conditions. This confirms a version of the ‘Hot Spots’ conjecture of J. Rauch for alcoves. We also prove the following



**Proposition 12.** *Let  $\mathcal{A}$  be the fundamental alcove of an affine Weyl group, and let the corresponding Weyl group have positive system  $\Phi^+$ . The function*

$$H(x) := \prod_{\alpha \in \Phi^+} \sin(\pi \langle x, \alpha \rangle) \quad (13)$$

*is an eigenfunction for the Laplacian with Dirichlet boundary conditions on  $\mathcal{A}$ . Since  $H$  is positive on  $\mathcal{A}$ , it is the principal eigenfunction. Further, each eigenfunction is divisible by  $H$  in the ring of trigonometric polynomials.*

### 3.6 The reflection principle and De Bruijn Formula

In this section we recall a reflection principle in the context of finite or affine reflection groups, and use it to deduce a generalisation of a formula of De Bruijn for evaluating multiple integrals involving determinants. For the proof of Theorem 13 we refer to [10] and references therein.

**Theorem 13.** *Let  $\mathbb{P}_x$  denote the law of a reflectable process  $X$  started from  $x \in F$ . Then for all measurable sets  $B \subset F$ ,*

$$\mathbb{P}_x[X_t \in B, T_\delta > t] = \sum_{\omega \in \mathcal{W}} \varepsilon(\omega) \mathbb{P}_x[X_t \in \omega B]. \quad (14)$$

We apply this result in the following propositions, whose applications include the evaluation of Selberg type integrals of eigenfunctions of the Dirichlet Laplacian on an alcove (see section 6).

Suppose  $I$  is consistent. For  $A \in \mathcal{I}$ , denote by  $W_A$  the group generated by the reflections  $s_\lambda$ ,  $\lambda \in A$ . Denote by  $F_A$  the fundamental region associated to  $A$ ,  $F_A = \{x \in V : \forall \lambda \in A, \lambda(x) > 0\}$ . Also, since  $\Phi = \Phi^+ \cup (-\Phi^+)$ , for  $\beta \in \Phi$  and  $B \subset \Phi$  we may define the absolute values

$$|\beta| = \begin{cases} \beta & : \beta \in \Phi^+ \\ -\beta & : -\beta \in \Phi^+ \end{cases}, \quad |B| = \{|\beta| : \beta \in B\}. \quad (15)$$

Assume that  $F_A$  is the fundamental region for the reflection group  $W_A$ , which is certainly the case if  $I$  is block-orthogonal or semi-orthogonal. Theorems 7 and 13 in the Brownian case give

**Proposition 14.** *If  $I$  is consistent and  $f : V \rightarrow \mathbb{R}$  is integrable, then*

$$\int_F \sum_{w \in \mathcal{W}} \varepsilon(w) f(wy) dy = \sum_{A \in \mathcal{I}} \varepsilon_A \sum_{w \in W_A} \varepsilon(w) \int_{F_A} f(wy) dy. \quad (16)$$

In many cases, if  $f$  factorises this formula may be expressed in terms of Pfaffians (see [8]); the type  $A$  case was first obtained by de Bruijn [6] using different methods. The next two results work out the corresponding results in the type  $\tilde{A}$  case.

**Proposition 15.** Let  $\mathcal{W} = \tilde{A}_{k-1}$  and let  $f(y_1, \dots, y_k) = f_1(y_1) \dots f_k(y_k)$  for integrable functions  $f_i : \mathbb{R} \rightarrow \mathbb{R}$ . If  $k$  is even then

$$\int_{\mathcal{A}} \sum_{\omega \in W_a} \varepsilon(\omega) f(\omega y) dy = \text{Pf}(J_{ij})_{i,j \in [k]}$$

where  $J_{ij} = \int (-1)^{\lfloor y-z \rfloor} f_i(y) f_j(z) dy dz$ .

**Proposition 16.** Under the conditions of Proposition 15, if  $k$  is odd then

$$\int_{\mathcal{A}} \sum_{\omega \in W_a} \varepsilon(\omega) f(\omega y) dy = \sum_{l=1}^k (-1)^{l+1} \int_{\mathbb{R}} f_l \text{Pf}(H_{ij})_{i,j \in [k] \setminus \{l\}}$$

if

$$\sum_{m=1}^{\infty} \int_{y-z \in (-\infty, -m) \cup (m, \infty)} |f_i(y) f_j(z)| dy dz < \infty,$$

where  $H_{ij} = \int \text{sgn}(y-z) f_i(y) f_j(z) dy dz + 2 \sum_{m=1}^{\infty} \int_{y-z \in (-\infty, -m) \cup (m, \infty)} \text{sgn}(y-z) f_i(y) f_j(z) dy dz$ .

## 4 Application to the different type cases

Throughout this section we will assume that  $X$  has independent components in orthogonal directions, to enable the writing of formula (5) in terms of Pfaffians.

### 4.1 The $\tilde{A}_{k-1}$ case, $k$ even


In this case,  $W$  is  $\mathfrak{S}_k$  acting on  $\mathbb{R}^k$  by permutation of the canonical basis vectors,  $V = \mathbb{R}^k$  or  $V = \{x \in \mathbb{R}^k : \sum_i x_i = 0\}$ ,  $\Phi^+ = \{e_i - e_j, 1 \leq i < j \leq k\}$ ,  $\Delta = \{e_i - e_{i+1}, 1 \leq i \leq k-1\}$ ,  $\tilde{\alpha} = e_1 - e_k$ ,  $\mathcal{A} = \{x \in V : 1 + x_k > x_1 > \dots > x_k\}$ ,  $\alpha^\vee = \alpha$  for  $\alpha \in \Phi$  and  $L = \{d \in \mathbb{Z}^k : \sum_{i=1}^k d_i = 0\}$ .

For even  $k = 2p$ , we take  $I = \{(e_{2i-1} - e_{2i}, 0), (-e_{2i-1} + e_{2i}, -1) ; 1 \leq i \leq p\}$ . Then  $I$  is consistent and block-orthogonal, and  $\mathcal{I}$  can be identified with the set  $P_2(k)$  of partitions of  $[k]$  as shown in the following example for  $k = 4$ . Under this identification, the sign  $\varepsilon_A$  is just the parity of the number  $c(\pi)$  of crossings.

Hence, the formula can be written as

$$\mathbb{P}_x(T > t) = \sum_{\pi \in P_2(k)} (-1)^{c(\pi)} \prod_{\{i < j\} \in \pi} \tilde{p}_{ij} = \text{Pf}(\tilde{p}_{ij})_{i,j \in [k]} \quad (17)$$

where  $\tilde{p}_{ij} = \mathbb{P}_x(T_{(e_i - e_j, 0), (-e_i + e_j, -1)} > t) = \mathbb{P}_x(\forall s \leq t, 0 < X_s^i - X_s^j < 1) = \phi(x_i - x_j, 2t)$  where  $\phi(x, t)$  is defined in (38).

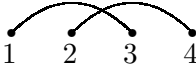


$$\pi = \{\{1, 4\}, \{2, 3\}\}$$

$$A = \{(e_1 - e_4, 0), (e_2 - e_3, 0),$$

$$(-e_1 + e_4, -1), (-e_2 + e_3, -1)\}$$

$$c(\pi) = 0$$

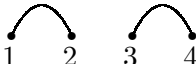


$$\pi = \{\{1, 3\}, \{2, 4\}\}$$

$$A = \{(e_1 - e_3, 0), (e_2 - e_4, 0),$$

$$(-e_1 + e_3, -1), (-e_2 + e_4, -1)\}$$

$$c(\pi) = 1$$



$$\pi = \{\{1, 2\}, \{3, 4\}\}$$

$$A = \{(e_1 - e_2, 0), (e_3 - e_4, 0),$$

$$(-e_1 + e_2, -1), (-e_3 + e_4, -1)\}$$

$$c(\pi) = 0$$

Figure 1: Pair partitions and their signs for  $\tilde{A}_3$ .

For odd  $k$ , we do not have a consistent subset as the sign  $\varepsilon_A$  is not well-defined. The difference between even and odd  $k$  can be seen directly at the level of pair partitions: interchanging 1 and  $k$  in the blocks of  $\pi \in P_2(k)$  (which corresponds to the reflection with respect to  $\{x_1 - x_k = 1\}$ , which is the affine hyperplane of the alcove) changes the sign of  $\pi$  if  $k$  is even while the sign is unaffected if  $k$  is odd. In this case (which includes, for example, the equilateral triangle in the case  $\tilde{A}_2$ ), we instead use Theorem 8.

## 4.2 The $\tilde{C}_k$ case

In this case,  $W$  is the group of signed permutations acting on  $V = \mathbb{R}^k$ ,  $\Delta = \{2e_k, e_i - e_{i+1}, 1 \leq i \leq k-1\}$ ,  $\tilde{\alpha} = 2e_1$ ,  $\mathcal{A} = \{x \in \mathbb{R}^k : 1/2 > x_1 > \dots > x_k > 0\}$  and  $L = \mathbb{Z}^k$ .

For even  $k = 2p$ , we take

$$I = \{(e_{2i-1} - e_{2i}, 0), (2e_{2i}, 0), (-2e_{2i-1}, -1); 1 \leq i \leq p\}.$$

For odd  $k = 2p + 1$ ,

$$I = \{(e_{2i-1} - e_{2i}, 0), (2e_{2i}, 0), (-2e_{2i-1}, -1), (2e_k, 0), (-2e_k, -1); 1 \leq i \leq p\}.$$

$I$  is semi-orthogonal and again,  $\mathcal{I}$  can be identified with  $P_2(k)$ ; the formula is

$$\mathbb{P}_x(T > t) = \sum_{\pi \in P_2(k)} (-1)^{c(\pi)} \check{p}_{s(\pi)} \prod_{\{i < j\} \in \pi} \check{p}_{ij} \quad (18)$$

where

$$\check{p}_{ij} = \mathbb{P}_x(T_{(e_i - e_j, 0), (2e_j, 0), (-2e_i, -1)} > t) = \mathbb{P}_x(\forall s \leq t, 1/2 > X_s^i > X_s^j > 0),$$

$$\check{p}_i = \mathbb{P}_x(T_{(2e_i, 0), (-2e_i, -1)} > t) = \mathbb{P}_x(\forall s \leq t, 1/2 > X_s^i > 0),$$

and  $s(\pi)$  is the singlet of  $\pi$ , the term  $\check{p}_{s(\pi)}$  being absent for even  $k$ .

Everything can be rewritten in terms of Pfaffians:

$$\mathbb{P}_x(T > t) = \begin{cases} \text{Pf } (\check{p}_{ij})_{i,j \in [k]} & \text{if } k \text{ is even,} \\ \sum_{l=1}^k (-1)^{l-1} \check{p}_l \text{Pf } (\check{p}_{ij})_{i,j \in [k] \setminus \{l\}} & \text{if } k \text{ is odd.} \end{cases} \quad (19)$$

**Remark 1.** *This formula can be obtained directly by applying the exit probability formula for the chamber of type  $C_k$  (which is the same as  $B_k$ ) to the Brownian motion killed when reaching  $1/2$ . But it was natural to include it in our framework.*

### 4.3 The $\tilde{B}_k$ case

$W$  is the group of signed permutations acting on  $V = \mathbb{R}^k$ ,  $\Delta = \{e_k, e_i - e_{i+1}, 1 \leq i \leq k-1\}$ ,  $\tilde{\alpha} = e_1 + e_2$ ,  $\mathcal{A} = \{x \in \mathbb{R}^k : x_1 > \dots > x_k > 0, x_1 + x_2 < 1\}$  and  $L = \{d \in \mathbb{Z}^k : \sum_i d_i \text{ is even}\}$ .

For even  $k = 2p$ , we take

$$I = \{(e_{2i-1} - e_{2i}, 0), (e_{2i}, 0), (-e_{2i-1} - e_{2i}, -1); 1 \leq i \leq p\}.$$

For odd  $k = 2p + 1$ ,

$$I = \{(e_{2i-1} - e_{2i}, 0), (e_{2i}, 0), (-e_{2i-1} - e_{2i}, -1), (e_k, 0), (-e_k, -1); 1 \leq i \leq p\}.$$

In this case,  $I$  is semi-orthogonal and the formula is:

$$\mathbb{P}_x(T > t) = \sum_{\pi \in P_2(k)} (-1)^{c(\pi)} \bar{p}_{s(\pi)} \prod_{\{i < j\} \in \pi} \bar{p}_{ij} \quad (20)$$

where

$$\begin{aligned} \bar{p}_{ij} &= \mathbb{P}_x(T_{(e_i - e_j, 0), (-e_i - e_j, -1), (e_j, 0)} > t) = \mathbb{P}_x(\forall s \leq t, 1 - X_s^j > X_s^i > X_s^j > 0), \\ \bar{p}_i &= \mathbb{P}_x(T_{(e_i, 0), (-e_i, -1)} > t) = \mathbb{P}_x(\forall s \leq t, 1 > X_s^i > 0) \end{aligned}$$

and  $s(\pi)$  denotes the singlet of  $\pi$ , the term  $\bar{p}_{s(\pi)}$  being absent for even  $k$ .

Everything can be rewritten in terms of Pfaffians:

$$\mathbb{P}_x(T > t) = \begin{cases} \text{Pf } (\bar{p}_{ij})_{i,j \in [k]} & \text{if } k \text{ is even,} \\ \sum_{l=1}^k (-1)^{l-1} \bar{p}_l \text{Pf } (\bar{p}_{ij})_{i,j \in [k] \setminus \{l\}} & \text{if } k \text{ is odd.} \end{cases} \quad (21)$$

#### 4.4 The $\tilde{D}_k$ case

$W$  is the group of evenly signed permutations acting on  $V = \mathbb{R}^k$ ,  $\Delta = \{e_i - e_{i+1}, e_{k-1} + e_k, 1 \leq i \leq k-1\}$ ,  $\tilde{\alpha} = e_1 + e_2$ ,  $\mathcal{A} = \{x \in \mathbb{R}^k : x_1 > \dots > x_{k-1} > |x_k|, x_1 + x_2 < 1\}$  and  $L = \{d \in \mathbb{Z}^k : \sum_i d_i \text{ is even}\}$ .

For even  $k = 2p$ , we take

$$I = \{(e_{2i-1} - e_{2i}, 0), (-e_{2i-1} + e_{2i}, -1), (e_{2i-1} + e_{2i}, 0), (-e_{2i-1} - e_{2i}, -1); 1 \leq i \leq p\}.$$

For odd  $k = 2p + 1$ ,

$$I = \{(e_{2i} - e_{2i+1}, 0), (-e_{2i} + e_{2i+1}, -1), (e_{2i} + e_{2i+1}, 0), (-e_{2i} - e_{2i+1}, -1); 1 \leq i \leq p\}.$$

$I$  is block-orthogonal and the formula then becomes:

$$\mathbb{P}_x(T > t) = \sum_{\pi \in P_2(k)} (-1)^{c(\pi)} \prod_{\{i < j\} \in \pi} \check{p}_{ij} \quad (22)$$

where

$$\begin{aligned} \check{p}_{ij} &= \mathbb{P}_x(T_{(e_i - e_j, 0), (-e_i + e_j, -1), (e_i + e_j, 0), (-e_i - e_j, -1)} > t) = \acute{p}_{ij} \grave{p}_{ij}, \\ \acute{p}_{ij} &= \mathbb{P}_x(\forall s \leq t, 1 > X_s^i - X_s^j > 0) = \phi(x_i - x_j, 2t), \\ \grave{p}_{ij} &= \mathbb{P}_x(\forall s \leq t, 1 > X_s^i + X_s^j > 0) = \phi(x_i + x_j, 2t) \end{aligned}$$

and  $\phi(x, t)$  is defined in (38). Everything can be rewritten in terms of Pfaffians:

$$\mathbb{P}_x(T > t) = \begin{cases} \text{Pf } (\check{p}_{ij})_{i,j \in [k]} & \text{if } k \text{ is even,} \\ \sum_{l=1}^k (-1)^{l-1} \text{Pf } (\check{p}_{ij})_{i,j \in [k] \setminus \{l\}} & \text{if } k \text{ is odd.} \end{cases} \quad (23)$$

#### 4.5 The $\tilde{G}_2$ case

Here,  $V = \{x \in \mathbb{R}^3, \sum_i x_i = 0\}$ ,  $\Phi^+ = \{e_3 - e_1, e_3 - e_2, e_1 - e_2, -2e_1 + e_2 + e_3, -2e_2 + e_1 + e_3, 2e_3 - e_1 - e_2\}$ ,  $\tilde{\alpha} = 2e_3 - e_1 - e_2$ ,  $\Delta = \{e_1 - e_2, -2e_1 + e_2 + e_3\}$  and  $L = \{d \in V : \forall i, 3d_i \in \mathbb{Z}\}$ .

We take  $I = \{(e_1 - e_2, 0), (-e_1 + e_2, -1), (2e_3 - e_1 - e_2, 0), (-2e_3 + e_1 + e_2, -1)\}$ , which is consistent and we can describe  $\mathcal{I}$  as  $\{I, A_1, A_2\}$  with  $A_1 = \{(e_3 - e_1, 0), (-e_3 + e_1, -1), (-2e_2 + e_1 + e_3, 0), (2e_2 - e_1 - e_3, -1)\}$ ,  $\varepsilon_{A_1} = -1$ ,

$A_2 = \{(e_3 - e_2, 0), (-e_3 + e_2, -1), (-2e_1 + e_2 + e_3, 0), (2e_1 - e_2 - e_3, -1)\}$ ,  
 $\varepsilon_{A_2} = 1$ .

In this case, the chamber  $\mathcal{A}$  is a triangle  $ABC$  with angles  $(\pi/2, \pi/3, \pi/6)$  as represented in Figure 2. If  $T_R$  denotes the exit time from the region  $R$  of the plane and  $\mathbb{P}(R) = \mathbb{P}_x(T_R > t)$ , then Theorem 7 in this case gives

$$\mathbb{P}(ABC) = \mathbb{P}(ADEC) - \mathbb{P}(FJCG) + \mathbb{P}(FHCI), \quad (24)$$

where  $ADEC$ ,  $FJCG$ ,  $FHCI$  are rectangles, as shown in Figure 11.2.

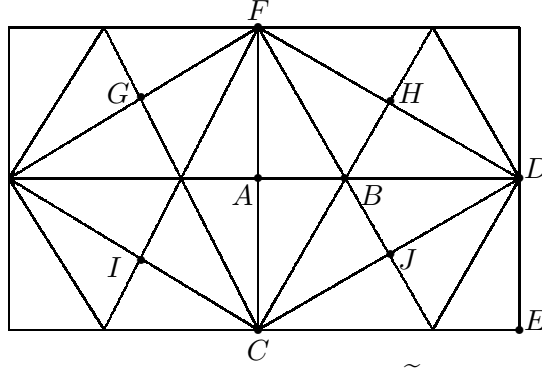


Figure 2: Tiling associated with  $\tilde{G}_2$

#### 4.6 The $\tilde{F}_4$ case

Recall that  $V = \mathbb{R}^4$ ,  $\Phi^+ = \{e_i \pm e_j, 1 \leq i < j \leq 4; e_i, 1 \leq i \leq 4; (e_1 \pm e_2 \pm e_3 \pm e_4)/2\}$ ,  $\Delta = \{e_2 - e_3, e_3 - e_4, e_4, (e_1 - e_2 - e_3 - e_4)/2\}$ ,  $\tilde{\alpha} = e_1 + e_2$  and  $L = \{d \in \mathbb{Z}^4 : \sum_i d_i \text{ is even}\}$ .

$I := \{(e_2 - e_3, 0), (-e_2 + e_3, -1), (e_1 - e_4, 0), (-e_1 + e_4, -1), (e_3, 0), (e_4, 0)\}$  turns out to be consistent and so Theorem 7 applies, although in this case it does not seem easy to give the formula in a compact way.

## 5 Proofs

### 5.1 Theorem 7

All the formalism of affine root systems has been set for the proofs in this section to be the same as those in [8]. Therefore, we only state the lemmas (without proofs) to show how they have to be modified in this context.

**Lemma 17.** *If  $I$  is consistent then for  $K \subset I$  and  $\lambda \in \delta \cap K^\perp$  we have  $s_\lambda \mathcal{L} = \mathcal{L}$ , where*

$$\mathcal{L} = \{A \in \mathcal{I} : K \subset A, \lambda \notin A\}.$$

**Lemma 18.** *Suppose condition (C3) is satisfied and that the function  $f : \mathcal{I} \rightarrow \mathbb{R}$  and the root  $\lambda \in \delta$  are such that  $f(A) = 0$  whenever  $\lambda \in A$ , and  $f(A) = f(s_\lambda A)$  whenever  $\lambda \notin A$ . Then  $\sum_{A \in \mathcal{I}} \varepsilon_A f(A) = 0$ .*

**Lemma 19.** *If  $I$  is consistent then we have:  $\sum_{A \in \mathcal{I}} \varepsilon_A = 1$ .*

**Proof of Theorem 7.** Appealing to Lemma 19 and the fact that  $T_\delta \leq T_A$  for all  $A \in \mathcal{I}$ , it is equivalent to prove  $\sum_{A \in \mathcal{I}} \varepsilon_A \mathbb{P}_x(T_A > t, T_\delta \leq t) = 0$  and therefore sufficient to prove  $\sum_{A \in \mathcal{I}} \varepsilon_A \mathbb{P}_x(T_A > t, T_\delta = T_\lambda \leq t) = 0$  for each  $\lambda \in \delta$ . Since  $X$  is reflectable,  $f(A) = \mathbb{P}_x(T_A > t, T_\delta = T_\lambda \leq t)$  satisfies the conditions of Lemma 18.  $\square$

## 5.2 Theorem 8

Before proving Theorem 8 we record some preliminary results. Since a consistent subset is available in the setting of the finite reflection group  $A_{k-1}$ , we work in this context. The definitions of  $V, \Delta, \tilde{\alpha}$  and  $\Phi^+$  when  $W = A_{k-1}$  have been given in section 4.1. It is proved in [8] that  $I = \{e_1 - e_2, e_3 - e_4, \dots, e_{k-2} - e_{k-1}\}$  is consistent and orthogonal. In the following we will make use of the notation introduced in sections 3.3-3.4 and (15). Also, for  $\beta \in \Phi^+$  define

$$\mathcal{L}_\beta = \{A \in \mathcal{I} : \beta \notin A\}.$$

**Lemma 20.**  *$A \mapsto |s_{\tilde{\alpha}} A|$  is a permutation of  $\mathcal{L}_{\tilde{\alpha}}$  and  $\varepsilon_{|s_{\tilde{\alpha}} A|} = (-1)^{|A \setminus \tilde{\alpha}^\perp|+1} \varepsilon_A$  for all  $A \in \mathcal{L}_{\tilde{\alpha}}$ .*

**Proof** Take  $A = \omega I \in \mathcal{L}_{\tilde{\alpha}}$ . Since the elements of  $A \setminus \tilde{\alpha}^\perp$  are orthogonal to each other so are those of  $s_{\tilde{\alpha}}(A \setminus \tilde{\alpha}^\perp)$  thus the product  $p := \prod_{\beta \in A \setminus \tilde{\alpha}^\perp} s_{s_{\tilde{\alpha}}(\beta)}$  is well-defined (and commutative). First, take  $\alpha \in A \cap \tilde{\alpha}^\perp$ . Then  $|s_{\tilde{\alpha}} \alpha| = |\alpha| = \alpha$ . If  $\beta \in A \setminus \tilde{\alpha}^\perp$  then  $\beta \neq \alpha$  hence  $\beta \perp \alpha$ . Together with  $\tilde{\alpha} \perp \alpha$ , we get  $s_{\tilde{\alpha}}(\beta) \perp \alpha$  and  $s_{s_{\tilde{\alpha}}(\beta)} s_{\tilde{\alpha}} \alpha = s_{s_{\tilde{\alpha}}(\beta)} \alpha = \alpha = |s_{\tilde{\alpha}} \alpha|$ . Thus,  $p s_{\tilde{\alpha}} \alpha = |s_{\tilde{\alpha}} \alpha|$ . Second, take  $\alpha \in A \setminus \tilde{\alpha}^\perp$ . Then  $s_{\tilde{\alpha}} \alpha \in -\Phi^+$  and  $|s_{\tilde{\alpha}} \alpha| = s_{s_{\tilde{\alpha}}(\alpha)} s_{\tilde{\alpha}} \alpha$ . For  $\beta \in A \setminus \tilde{\alpha}^\perp$  and  $\beta \neq \alpha$  we have  $\beta \perp -\alpha$  so  $s_{\tilde{\alpha}} \beta \perp -s_{\tilde{\alpha}} \alpha = s_{s_{\tilde{\alpha}}(\alpha)} s_{\tilde{\alpha}} \alpha$ . Therefore  $p s_{\tilde{\alpha}} \alpha = s_{s_{\tilde{\alpha}}(\alpha)} s_{\tilde{\alpha}} \alpha = |s_{\tilde{\alpha}} \alpha|$ . We have proved that  $|s_{\tilde{\alpha}} A| = p s_{\tilde{\alpha}} A = p s_{\tilde{\alpha}} \omega I$ . Together with  $|s_{\tilde{\alpha}} A| \subset \Phi^+$ , this yields  $|s_{\tilde{\alpha}} A| \in \mathcal{I}$  and  $\varepsilon_{|s_{\tilde{\alpha}} A|} = (-1)^{|A \setminus \tilde{\alpha}^\perp|+1} \varepsilon_A$ . Moreover  $\tilde{\alpha} \notin |s_{\tilde{\alpha}} A|$  since  $\tilde{\alpha} \notin A$ . Consequently  $|s_{\tilde{\alpha}} A| \in \mathcal{L}_{\tilde{\alpha}}$ . It remains to observe that  $A \mapsto |s_{\tilde{\alpha}} A|$  is an involution hence a bijection.  $\square$

Observing that  $\langle e_1 - e_k, e_1 - e_j \rangle = \langle e_1 - e_k, e_i - e_k \rangle = 1$  for  $1 < i, j < k$  gives

**Lemma 21.** *For all  $\beta \in \Phi^+ \setminus (\tilde{\alpha} \cup \tilde{\alpha}^\perp)$  we have  $\langle \tilde{\alpha}, \beta \rangle = 1$ .*

Also, calculations such as

$$\begin{aligned}\varepsilon_{s_\alpha v}^A &= (-1)^{\#\{\beta \in A : \langle s_\alpha v, \beta \rangle > 0\}} = (-1)^{\#\{\beta \in s_\alpha A : \langle s_\alpha v, s_\alpha \beta \rangle > 0\}} \\ &= (-1)^{\#\{\beta \in s_\alpha A : \langle v, \beta \rangle > 0\}} = \varepsilon_v^{s_\alpha A}\end{aligned}$$

establish

**Lemma 22.** *For all  $\alpha \in \Phi$ ,  $A \in \mathcal{O}(\Phi)$  and  $v \in E_A$  we have*

$$s_\alpha E_A = E_{s_\alpha A}, \quad \varepsilon_{s_\alpha v}^A = \varepsilon_v^{s_\alpha A} \text{ and } |s_\alpha v|_A = |v|_{s_\alpha A}.$$

**Proposition 23.** (i)  $\sum_{A \in \mathcal{I}} \sum_{k \in \mathbb{N}} \sum_{\substack{v \in E_A \\ |v|_A = k}} \varepsilon_A \varepsilon_v^A = 1.$

(ii) Suppose  $f : \mathcal{I} \times V \rightarrow \mathbb{R}$  is such that  $f(A, v) = f(|s_{\tilde{\alpha}} A|, p_{s_{\tilde{\alpha}} A}(s_{\tilde{\alpha}, 1} v))$  whenever  $\tilde{\alpha} \notin A$  ( $p_B$  is the orthogonal projection on  $\text{Span}(B)$ ) and  $f$  is sufficiently decreasing in the second variable (see the precise condition (28) in the proof). Then  $\sum_{A \in \mathcal{L}_{\tilde{\alpha}}} \sum_{k \in \mathbb{N}} \sum_{\substack{v \in E_A \\ |v|_A = k}} \varepsilon_A \varepsilon_v^A f(A, v)$  converges and its sum is zero.

(iii) If  $f : \mathcal{I} \times V \rightarrow \mathbb{R}$  and  $\alpha \in \Delta$  are such that  $f(A, v) = f(s_\alpha A, s_\alpha v)$  whenever  $\alpha \notin A$ , then  $\sum_{A \in \mathcal{L}_\alpha} \sum_{k \in \mathbb{N}} \sum_{\substack{v \in E_A \\ |v|_A = k}} \varepsilon_A \varepsilon_v^A f(A, v)$  converges and its sum is zero.

**Proof** (i) For  $A \in \mathcal{O}(\Phi)$  and  $\alpha \in A$  define

$$S(A, k) = \sum_{\substack{v \in E_A \\ |v|_A = k}} \varepsilon_v^A \quad \text{and} \quad S'(A, \alpha, k) = \sum_{\substack{v \in E_A \\ |v|_A = k}} \mathbf{1}_{v \notin \alpha^\perp} \varepsilon_v^A,$$

where  $\mathbf{1}$  is the indicator function. Since  $\varepsilon_0^A = 1$  for all  $A \in \mathcal{I}$  and  $\sum_{A \in \mathcal{I}} \varepsilon_A = 1$  by Lemma 19, we have

$$\sum_{A \in \mathcal{I}} \sum_{k \in \mathbb{N}} \sum_{\substack{v \in E_A \\ |v|_A = k}} \varepsilon_A \varepsilon_v^A = 1 + \sum_{A \in \mathcal{I}} \varepsilon_A \sum_{k \geq 1} S(A, k). \quad (25)$$

If  $u \notin \alpha^\perp$ ,  $\varepsilon_{s_\alpha u}^A = \varepsilon_u^{s_\alpha A} = \varepsilon_u^{A \setminus \{\alpha\} \cup \{-\alpha\}} = -\varepsilon_u^A$ . Thus, setting  $v = s_\alpha u$  in  $S'(A, \alpha, k)$  and using  $s_\alpha E_A = E_{s_\alpha A} = E_A$ ,  $|s_\alpha u|_A = |u|_{s_\alpha A} = |u|_A$ ,  $\mathbf{1}_{s_\alpha u \notin \alpha^\perp} = \mathbf{1}_{u \notin \alpha^\perp}$ , we get  $S'(A, \alpha, k) = \sum_{\substack{u \in E_A \\ |u|_A = k}} \mathbf{1}_{u \notin \alpha^\perp} \varepsilon_{s_\alpha u}^A = -S'(A, \alpha, k) = 0$ . Therefore

$$S(A, k) = \sum_{\substack{v \in E_A \cap \alpha^\perp \\ |v|_A = k}} \varepsilon_v^A = \sum_{\substack{v \in E_{A \setminus \{\alpha\}} \\ |v|_{A \setminus \{\alpha\}} = k}} \varepsilon_v^A = S(A \setminus \{\alpha\}, k).$$

By iteration  $S(A, k) = S(\emptyset, k)$ , which is an empty sum (since  $E_\emptyset = \{0\}$  and  $k \geq 1$ ) hence null.



(ii) Take  $A \in \mathcal{L}_{\tilde{\alpha}}$  and  $u \in E_A$ . Now  $\varepsilon_{s_{\tilde{\alpha},1}u}^{|s_{\tilde{\alpha}}A|} = (-1)^{\#\{\beta \in |s_{\tilde{\alpha}}A| : \langle s_{\tilde{\alpha},1}u, \beta \rangle > 0\}}$  and  $s_{\tilde{\alpha},1}u = s_{\tilde{\alpha}}u + \tilde{\alpha}$ ; therefore if  $\beta \in |s_{\tilde{\alpha}}A| \setminus \tilde{\alpha}^\perp$  then writing  $\gamma = -s_{\tilde{\alpha}}\beta \in A \setminus \tilde{\alpha}^\perp$  and applying Lemma 21 we have  $(\langle s_{\tilde{\alpha},1}u, \beta \rangle > 0 \iff \langle u, \gamma \rangle < 1)$ . Also, if  $\beta \in |s_{\tilde{\alpha}}A| \cap \tilde{\alpha}^\perp = A \cap \tilde{\alpha}^\perp$  then  $(\langle s_{\tilde{\alpha},1}u, \beta \rangle > 0 \iff \langle u, \beta \rangle > 0)$ . We conclude that

$$\begin{aligned} \varepsilon_{s_{\tilde{\alpha},1}u}^{|s_{\tilde{\alpha}}A|} &= (-1)^{\#\{\gamma \in A \setminus \tilde{\alpha}^\perp : \langle u, \gamma \rangle < 1\} + \#\{\beta \in A \cap \tilde{\alpha}^\perp : \langle u, \beta \rangle > 0\}} \\ &= (-1)^{\#\{\gamma \in A \setminus \tilde{\alpha}^\perp : \langle u, \gamma \rangle < 1\} + \#\{\beta \in A \setminus \tilde{\alpha}^\perp : \langle u, \beta \rangle > 0\}} \varepsilon_u^A = (-1)^{|A \setminus \tilde{\alpha}^\perp|} \varepsilon_u^A. \end{aligned}$$

Since  $\varepsilon_{|s_{\tilde{\alpha}}A|} = (-1)^{|A \setminus \tilde{\alpha}^\perp|+1} \varepsilon_A$  by Lemma 20, we have

$$\varepsilon_{|s_{\tilde{\alpha}}A|} \varepsilon_{s_{\tilde{\alpha},1}u}^{|s_{\tilde{\alpha}}A|} f(|s_{\tilde{\alpha}}A|, p_{s_{\tilde{\alpha}}A}(s_{\tilde{\alpha},1}u)) = -\varepsilon_A \varepsilon_u^A f(A, u). \quad (26)$$

For  $K \in \mathbb{N} = \{0, 1, 2, \dots\}$ , set

$$S_K = \sum_{A \in \mathcal{L}_{\tilde{\alpha}}} \sum_{k=0}^K \sum_{\substack{v \in E_A \\ |v|_A = k}} \varepsilon_A \varepsilon_v^A f(A, v).$$

Using the permutation  $A \mapsto |s_{\tilde{\alpha}}A|$  of  $\mathcal{L}_{\tilde{\alpha}}$  from Lemma 20 and since both  $E_{|B|} = E_B$  and  $|v|_{|B|} = |v|_B$  for  $B \subset \mathcal{O}(\Phi)$ , we get

$$S_K = \sum_{A \in \mathcal{L}_{\tilde{\alpha}}} \sum_{k=0}^K \sum_{\substack{v \in E_{s_{\tilde{\alpha}}A} \\ |v|_{s_{\tilde{\alpha}}A} = k}} \varepsilon_{|s_{\tilde{\alpha}}A|} \varepsilon_v^{|s_{\tilde{\alpha}}A|} f(|s_{\tilde{\alpha}}A|, v).$$

For  $A \in \mathcal{L}_{\tilde{\alpha}}$  and  $u \in E_A$ , define  $g_A(u) = p_{s_{\tilde{\alpha}}A}(s_{\tilde{\alpha},1}u) = s_{\tilde{\alpha}}u + p_{s_{\tilde{\alpha}}A}(\tilde{\alpha})$ . Then  $g_A(u) \in \text{Span}(s_{\tilde{\alpha}}A)$  and for all  $\beta \in A$ ,

$$\langle g_A(u), s_{\tilde{\alpha}}\beta \rangle = \langle s_{\tilde{\alpha},1}u, s_{\tilde{\alpha}}\beta \rangle = \langle u, \beta \rangle - \langle \tilde{\alpha}, \beta \rangle \in \mathbb{Z}$$

since  $u \in E_A$  and  $\langle \tilde{\alpha}, \beta \rangle \in \{0, 1\}$  (Lemma 21). This proves that  $g_A(u) \in E_{s_{\tilde{\alpha}}A}$  and  $|g_A(u)|_{s_{\tilde{\alpha}}A} = |u|_A + \eta_A(u)$  where  $\eta_A(u) \in \{-1, 0, 1\}$ . Then,  $g_A : E_A \rightarrow E_{s_{\tilde{\alpha}}A}$  is easily seen to be a bijection (check that  $g_A^{-1}(v) = p_A(s_{\tilde{\alpha},1}v)$ ). Using this bijection as well as (26), we obtain

$$S_K = - \sum_{A \in \mathcal{L}_{\tilde{\alpha}}} \sum_{k=0}^K \sum_{\substack{u \in E_A \\ |u|_A + \eta_A(u) = k}} \varepsilon_A \varepsilon_u^A f(A, u). \quad (27)$$

Now, for  $i \in \{-1, 0, 1\}$ , let  $S_i(k) = \sum_{A \in \mathcal{L}_{\tilde{\alpha}}} \sum_{\substack{u \in E_A \\ |u|_A = k, \eta_A(u) = i}} \varepsilon_A \varepsilon_u^A f(A, u)$ .

Then (27) reads

$$S_K = - \sum_{k=0}^K (S_0(k) + S_1(k-1) + S_{-1}(k+1)).$$

Since  $S_K = \sum_{k=0}^K (S_0(k) + S_1(k) + S_{-1}(k))$  by definition, we get

$$2S_K = -S_1(-1) + S_1(K) + S_{-1}(0) - S_{-1}(K+1).$$

Now,  $S_1(-1)$  and  $S_{-1}(0)$  are empty sums hence null. The requirement on  $f$  is

$$\lim_{k \rightarrow \infty} \sum_{\substack{A \in \mathcal{I}, u \in E_A \\ |u|_A = k}} |f(A, u)| = 0, \quad (28)$$

which clearly implies  $\lim_{K \rightarrow \infty} S_i(K) = 0$  and consequently  $\lim_{K \rightarrow \infty} S_K = 0$ .

(iii) Since  $s_\alpha$  is a permutation of  $\mathcal{L}_\alpha$  (Lemma 17),

$$\begin{aligned} U_K &:= \sum_{A \in \mathcal{L}_\alpha} \sum_{k=0}^K \sum_{\substack{v \in E_A \\ |v|_A = k}} \varepsilon_A \varepsilon_v^A f(A, v) = \sum_{A \in \mathcal{L}_\alpha} \sum_{k=0}^K \sum_{\substack{v \in E_{s_\alpha A} \\ |v|_{s_\alpha A} = k}} \varepsilon_{s_\alpha A} \varepsilon_v^{s_\alpha A} f(s_\alpha A, v) \\ &= \sum_{A \in \mathcal{L}_\alpha} \sum_{k=0}^K \sum_{\substack{u \in s_\alpha E_{s_\alpha A} \\ |s_\alpha u|_{s_\alpha A} = k}} \varepsilon_{s_\alpha A} \varepsilon_{s_\alpha u}^{s_\alpha A} f(s_\alpha A, s_\alpha u) = -U_K, \end{aligned}$$

where the third equality follows from setting  $u = s_\alpha v$  and the fourth follows from Lemma 22, the given property of  $f$  and  $\varepsilon_{s_\alpha A} = -\varepsilon_A$ . Thus, all partial sums  $U_K$  are zero.  $\square$

### Proof of Theorem 8

From (i) of Proposition 23, the theorem is equivalent to

$$\sum_{A \in \mathcal{I}} \sum_{k \in \mathbb{N}} \sum_{\substack{v \in E_A \\ |v|_A = k}} \varepsilon_A \varepsilon_v^A \left( \mathbb{P}_x[T_{A,v} > t] - \mathbb{P}_x[\tilde{T} > t] \right) = 0.$$

For  $A \in \mathcal{I}$ ,  $v \in E_A$  and  $\beta \in A$  we have  $\langle v, \beta \rangle \in \mathbb{Z}$  hence  $\langle v, \beta \rangle \notin (0, 1)$ . Thus,  $\tilde{T} \leq T_{\beta,v}$  and so  $\tilde{T} \leq T_{A,v}$ . This implies

$$\begin{aligned} \mathbb{P}_x[T_{A,v} > t] - \mathbb{P}_x[\tilde{T} > t] &= \mathbb{P}_x[T_{A,v} > t, \tilde{T} \leq t] \\ &= \sum_{\lambda \in \Delta_a} \mathbb{P}_x[T_{A,v} > t, \tilde{T} = T_\lambda \leq t]. \quad (29) \end{aligned}$$

(If the events in (29) are not disjoint (up to a set of probability zero) we may easily redefine the  $T_\lambda$  to make them disjoint, without affecting the following reflection argument.) Now fix  $\lambda = (\alpha, n) \in \{\Delta \times \{0\}\} \cup \{(\tilde{\alpha}, 1)\}$  (this set is more convenient than  $\Delta_a$  since we have  $(\tilde{\alpha}, 1)$  instead of  $(-\tilde{\alpha}, -1)$ ). We will prove that

$$\sum_{A \in \mathcal{I}} \sum_{k \in \mathbb{N}} \sum_{\substack{v \in E_A \\ |v|_A = k}} \varepsilon_A \varepsilon_v^A \mathbb{P}_x[T_{A,v} > t, \tilde{T} = T_\lambda \leq t] = 0.$$

Since  $\mathbb{P}_x[T_{A,v} > t, \tilde{T} = T_\lambda \leq t] = \mathbb{P}_x[\tilde{T} = T_\lambda \leq t] - \mathbb{P}_x[T_{A,v} \leq t, \tilde{T} = T_\lambda \leq t]$  and using (i) of Proposition 23 again, this is equivalent to

$$S := \sum_{A \in \mathcal{I}} \sum_{k \in \mathbb{N}} \sum_{\substack{v \in E_A \\ |v|_A = k}} \varepsilon_A \varepsilon_v^A f(A, v) = \mathbb{P}_x[\tilde{T} = T_\lambda \leq t],$$

where  $f(A, v) = \mathbb{P}_x[T_{A,v} \leq t, \tilde{T} = T_\lambda \leq t]$ . We first prove that

$$f(A, v) = f(s_\alpha A, s_\lambda v). \quad (30)$$

Since  $f(A, v) = \mathbb{P}_x[\tilde{T} = T_\lambda \leq t] - g(A, v)$  where  $g(A, v) = \mathbb{P}_x[T_{A,v} > t, \tilde{T} = T_\lambda \leq t]$ , it is enough to prove  $g(A, v) = g(s_\alpha A, s_\lambda v)$ . We define  $\hat{X}_u = X_u \mathbf{1}_{u \leq T_\lambda} + s_\lambda X_u \mathbf{1}_{u > T_\lambda}$  and use obvious ‘hat notations’ for stopping times associated with  $\hat{X}$ . The reflectable process  $X$  has the same law as  $\hat{X}$  so that  $g(A, v) = \mathbb{P}_x[\hat{T}_{A,v} > t, \hat{\tilde{T}} = \hat{T}_\lambda \leq t]$ . Since  $X$  and  $\hat{X}$  coincide before  $T_\lambda = \hat{T}_\lambda$ , we have  $\hat{\tilde{T}} = \tilde{T}$ . Together with  $\hat{T}_{A,v} \mathbf{1}_{\hat{T}_{A,v} > T_\lambda} = T_{s_\alpha A, s_\lambda v} \mathbf{1}_{T_{s_\alpha A, s_\lambda v} > T_\lambda}$ , this yields

$$g(A, v) = \mathbb{P}_x[T_{s_\alpha A, s_\lambda v} > t, \tilde{T} = T_\lambda \leq t] = g(s_\alpha A, s_\lambda v),$$

which proves the claim.

In addition to the equality  $f(A, v) = f(|A|, p_A v)$ , equation (30) ensures that  $f$  has the relevant property for Proposition 23 to yield

$$\sum_{A \in \mathcal{L}_\alpha} \sum_{k \in \mathbb{N}} \sum_{\substack{v \in E_A \\ |v|_A = k}} \varepsilon_A \varepsilon_v^A f(A, v) = 0$$

so that  $S = \sum_{\substack{A \in \mathcal{I} \\ \alpha \in A}} \sum_{k \in \mathbb{N}} \sum_{\substack{v \in E_A \\ |v|_A = k}} \varepsilon_A \varepsilon_v^A f(A, v)$ . If  $\alpha \in A$  then  $f(A, v) = f(A, s_\lambda v)$  (thanks to (30)) and if  $\lambda(v) \neq 0$  then  $\varepsilon_v^A = -\varepsilon_{s_\lambda v}^A$ . Then as in the proof of Proposition 23(ii) we can use the bijection  $v \mapsto s_\lambda v$  to remove cancelling pairs and appeal to property (28) to conclude that

$$\sum_{k \in \mathbb{N}} \sum_{\substack{v \in E_A \\ |v|_A = k}} \mathbf{1}_{\lambda(v) \neq 0} \varepsilon_v^A f(A, v) = 0$$

so that  $S = \sum_{\substack{A \in \mathcal{I} \\ \alpha \in A}} \varepsilon_A S(A)$ , where  $S(A) := \sum_{k \in \mathbb{N}} \sum_{\substack{v \in E_A \\ |v|_A = k}} \varepsilon_v^A \mathbf{1}_{\lambda(v)=0} f(A, v)$ .

If  $\alpha \in A$  and  $\lambda(v) = 0$  then  $f(A, v) = \mathbb{P}_x[\tilde{T} = T_\lambda \leq t]$  and

$$S(A) = \mathbb{P}_x[\tilde{T} = T_\lambda \leq t] \sum_{k \in \mathbb{N}} \sum_{\substack{v \in E_A \\ |v|_A = k}} \varepsilon_v^A \mathbf{1}_{\lambda(v)=0}.$$

For  $\beta \in A \setminus \{\alpha\}$ , the bijection  $v \mapsto s_\beta v$  flips the sign  $\varepsilon_v^A$  creating pair cancellations for the terms with  $v$  not orthogonal to  $\beta$ . Repeating this for all

$\beta \neq \alpha$  as in the proof of Proposition 23(i), we are left only with that  $v$  which is a multiple of  $\alpha$  such that  $\lambda(v) = 0$ , i.e.  $v = n\alpha/2$  : we have

$$S(A) = \varepsilon_{n\alpha/2}^A \mathbb{P}_x[\tilde{T} = T_\lambda \leq t].$$

It remains only to show that  $\sum_{\substack{A \in \mathcal{I} \\ \alpha \in A}} \varepsilon_A \varepsilon_{n\alpha/2}^A = 1$ . When  $\alpha \in \Delta$  this follows from the proof of Lemma 19, which can be found in [8]; for  $\alpha = \tilde{\alpha}$ , observe that  $\varepsilon_{\tilde{\alpha}/2}^A = -1$  if  $\tilde{\alpha} \in A$ . Identifying  $A \in \mathcal{I}$  with  $\pi \in P_2(k)$  as in section 4.1, we have  $(\tilde{\alpha} \in A \iff \{1, k\} \in \pi)$ . Now  $\{1, k\}$  crosses the pair containing 0, and no other pair. It follows that  $c(\pi) = 1 + c(\pi \setminus \{1, k\})$ , so

$$\sum_{\substack{A \in \mathcal{I} \\ \tilde{\alpha} \in A}} \varepsilon_A = \sum_{\substack{\pi \in P_2(k) \\ \{1, k\} \in \pi}} (-1)^{c(\pi)} = - \sum_{\pi \in P_2(k-2)} (-1)^{c(\pi)} = -1$$

by Lemma 19. □

### 5.3 Proposition 10

**Lemma 24.** *For  $A \in \mathcal{O}(\Phi^+)$ , if projections of  $X$  in orthogonal directions are independent then for  $x \in \mathcal{A}$ ,*

$$\sum_{k \in \mathbb{N}} \sum_{\substack{v \in E_A \\ |v|_A = k}} \varepsilon_v^A \mathbb{P}_x[T_{A,v} > t] = \prod_{\beta \in A} \sum_{n \in \mathbb{N}} \sum_{\substack{k \in \mathbb{Z} \\ |k| = n}} \sigma(k) \mathbb{P}_x[T_{(\beta,k)} > t], \quad (31)$$

where  $\sigma(k) = -1$  if  $k > 0$  and  $\sigma(k) = 1$  otherwise, if these sums converge.

**Proof** Set  $A = \{\beta_1, \dots, \beta_p\}$ . Rewriting and expanding the respective partial sums gives, for  $N \in \mathbb{N}$ ,

$$\prod_{i=1}^p \sum_{n=0}^N \sum_{\substack{k \in \mathbb{Z} \\ |k| = n}} \sigma(k) \mathbb{P}_x[T_{(\beta_i, k)} > t] = \sum_{n=0}^N \sum_{\substack{\vec{k} = (k_1, \dots, k_p) \in \mathbb{Z}^p \\ |\vec{k}|_\infty = n}} \prod_{i=1}^p \sigma(k_i) \mathbb{P}_x[T_{(\beta_i, k_i)} > t].$$

Now,  $\vec{k} = (k_1, \dots, k_p) \mapsto v = \frac{1}{2} \sum_{i=1}^p k_i \beta_i$  is a bijection from  $\mathbb{Z}^p$  to  $E_A$  satisfying  $\langle v, \beta_i \rangle = k_i$  so that  $T_{(\beta_i, k_i)} = T_{\beta_i, v}$ ,  $|v|_A = |\vec{k}|_\infty$  and  $\varepsilon_v^A = \prod_{i=1}^p \sigma(k_i)$ . By independence  $\prod_{i=1}^p \mathbb{P}_x[T_{(\beta_i, k_i)} > t] = \mathbb{P}_x[\min_i T_{(\beta_i, k_i)} > t] = \mathbb{P}_x[T_{A,v} > t]$ , and letting  $N \rightarrow \infty$  concludes the proof. □

**Lemma 25.** *If  $X$  is reflectable then for  $x \in \mathcal{A}$ ,*

$$\mathbb{P}_x[T_\beta \wedge T_{(\beta,1)} > t] + 2\mathbb{P}_x[T_\beta > T_{(\beta,1)} \leq t] = \sum_{n \in \mathbb{N}} \sum_{\substack{k \in \mathbb{Z} \\ |k| = n}} \sigma(k) \mathbb{P}_x[T_{(\beta,k)} > t].$$

**Proof** Let

$$\begin{aligned}
S1 &= \sum_{k=1}^{\infty} (\mathbb{P}_x[T_{(\beta,-k)} > t, T_{\beta} \wedge T_{(\beta,1)} > t] - \mathbb{P}_x[T_{(\beta,k)} > t, T_{\beta} \wedge T_{(\beta,1)} > t]) \\
S2 &= \sum_{k=1}^{\infty} (\mathbb{P}_x[T_{(\beta,-k)} > t, T_{(\beta,1)} > T_{\beta} \leq t] - \mathbb{P}_x[T_{(\beta,k)} > t, T_{(\beta,1)} > T_{\beta} \leq t]) \\
S3 &= \sum_{k=1}^{\infty} (\mathbb{P}_x[T_{(\beta,-k)} > t, T_{\beta} > T_{(\beta,1)} \leq t] - \mathbb{P}_x[T_{(\beta,k)} > t, T_{\beta} > T_{(\beta,1)} \leq t])
\end{aligned}$$

Then the implication  $(T_{\beta} \wedge T_{(\beta,1)} > t \Rightarrow \forall k, T_{(\beta,k)} > t)$  shows that all summands in  $S1$  are 0. For  $S3$  set  $a_k = \mathbb{P}_x[T_{(\beta,-k)} > t, T_{\beta} > T_{(\beta,1)} \leq t]$  and  $b_k = \mathbb{P}_x[T_{(\beta,k)} > t, T_{\beta} > T_{(\beta,1)} \leq t]$ . Set  $X'_u = X_u \mathbf{1}_{u \leq T_{(\beta,1)}} + s_{\beta,1} X_u \mathbf{1}_{u > T_{(\beta,1)}}$ . Then  $X$  and  $X'$  have the same law so  $a_k = \mathbb{P}_x[T'_{(\beta,-k)} > t, T'_{\beta} > T'_{(\beta,1)} \leq t]$ . For  $k \in \mathbb{Z}$ , the definition of  $X'$  gives

$$T'_{(\beta,-k)} = T_{(\beta,-k)} \mathbf{1}_{T_{(\beta,-k)} \leq T_{(\beta,1)}} + (T_{(\beta,2+k)} \circ \theta_{T_{(\beta,1)}} + T_{(\beta,1)}) \mathbf{1}_{T_{(\beta,-k)} > T_{(\beta,1)}} \quad (32)$$

where  $\theta$  is the shift operator. With  $k = -1$  this gives  $T'_{(\beta,1)} = T_{(\beta,1)}$ . With  $k = 0$  we get  $\{T'_{\beta} > T_{(\beta,1)} \leq t\} = \{T_{\beta} > T_{(\beta,1)} \leq t\}$  and for all  $k$ ,

$$a_k = \mathbb{P}_x[T'_{(\beta,-k)} > t, T_{\beta} > T_{(\beta,1)} \leq t]. \quad (33)$$

If  $k \geq 1$  and  $T_{\beta} > T_{(\beta,1)}$  then  $T_{(\beta,-k)} \geq T_{\beta} > T_{(\beta,1)}$ , so (32) gives  $T'_{(\beta,-k)} = T_{(\beta,2+k)} \circ \theta_{T_{(\beta,1)}} + T_{(\beta,1)}$ . So (33) becomes

$$a_k = \mathbb{P}_x[T_{(\beta,2+k)} \circ \theta_{T_{(\beta,1)}} + T_{(\beta,1)} > t, T_{\beta} > T_{(\beta,1)} \leq t].$$

For  $k \geq 0$ ,  $T_{(\beta,2+k)} > T_{(\beta,1)}$  so  $T_{(\beta,2+k)} = T_{(\beta,2+k)} \circ \theta_{T_{(\beta,1)}} + T_{(\beta,1)}$  and

$$a_k = \mathbb{P}_x[T_{(\beta,2+k)} > t, T_{\beta} > T_{(\beta,1)} \leq t] = b_{2+k}.$$

In this way we get  $S3 = 2 \lim_{k \rightarrow +\infty} a_k - b_1 - b_2$ . Now  $b_1 = 0$ ,  $b_2 = a_0$  and since  $\{X(s) : 0 \leq s \leq t\}$  is almost surely bounded we have  $\lim_{k \rightarrow +\infty} a_k = \mathbb{P}_x[T_{\beta} > T_{(\beta,1)} \leq t]$  so that

$$S3 = 2\mathbb{P}_x[T_{\beta} > T_{(\beta,1)} \leq t] - \mathbb{P}_x[T_{(\beta,1)} \leq t, T_{\beta} > t].$$

The same line of reasoning gives  $S2 = 0$ . Finally observe that

$$\sum_{k=1}^{\infty} (\mathbb{P}_x[T_{(\beta,-k)} > t] - \mathbb{P}_x[T_{(\beta,k)} > t]) = S1 + S2 + S3.$$

□

**Proof of Proposition 10** Apply Lemmas 24 and 25 to Theorem 8.

## 5.4 Consistency in the different type cases

### 5.4.1 $\tilde{A}_{k-1}$ , $k$ even

Let us first determine  $\mathcal{I}$ . If  $w_a = \tau(d)\sigma \in W_a^I$ , then

$$w_a\{(e_{2i-1} - e_{2i}, 0), (-e_{2i-1} + e_{2i}, -1)\} =$$

$$\{(e_{\sigma(2i-1)} - e_{\sigma(2i)}, n), (-e_{\sigma(2i-1)} + e_{\sigma(2i)}, -1 - n)\},$$

where  $n = d_{\sigma(2i-1)} - d_{\sigma(2i)}$ . Thus,  $n \leq 0$  and  $-1 - n \leq 0$ , ie  $n \in \{0, -1\}$ . If  $n = 0$ ,  $d_{\sigma(2i-1)} = d_{\sigma(2i)}$  and  $\sigma(2i-1) < \sigma(2i)$ . If  $n = -1$ ,  $d_{\sigma(2i-1)} = d_{\sigma(2i)} - 1$  and  $\sigma(2i-1) > \sigma(2i)$ . In any case,

$$w_a\{(e_{2i-1} - e_{2i}, 0), (-e_{2i-1} + e_{2i}, -1)\} =$$

$$\{(e_{\min(\sigma(2i-1), \sigma(2i))} - e_{\max(\sigma(2i-1), \sigma(2i))}, 0), (-e_{\min(\sigma(2i-1), \sigma(2i))} + e_{\max(\sigma(2i-1), \sigma(2i))}, -1)\}.$$

Thus, we identify  $\pi = \{\{i_l < j_l\}, 1 \leq l \leq p\} \in P_2(k)$  and  $A = \{(e_{i_l} - e_{j_l}, 0), (-e_{i_l} + e_{j_l}, -1); 1 \leq l \leq p\} \in \mathcal{I}$ . Then we take  $J_a = \{(e_{2i-1} - e_{2i}, 0); 1 \leq i \leq p\} \in \mathcal{O}(\Delta_a)$ . From the previous description of  $\mathcal{I}$ , (C1) and (C3) are obvious. Now it is clear that

$$U_a = \{\tau(d)\sigma : \sigma \text{ permutes sets } \{1, 2\}, \{3, 4\}, \dots, \{k-1, k\} \text{ and } \forall i \leq p,$$

$$(d_{\sigma(2i-1)} = d_{\sigma(2i)}, \sigma(2i-1) < \sigma(2i)) \text{ or } (d_{\sigma(2i-1)} = d_{\sigma(2i)} - 1, \sigma(2i-1) > \sigma(2i))\}.$$

Thus if  $\tau(d)\sigma \in U_a$  we can write  $\sigma = \sigma_1\sigma_2$ , where  $\sigma_2$  permutes pairs  $(1, 2), \dots, (k-1, k)$  and  $\sigma_1$  is the product of the transpositions  $(\sigma(2i-1), \sigma(2i))$  for which  $d_{\sigma(2i-1)} = d_{\sigma(2i)} - 1$ . Then  $\varepsilon(\sigma_2) = 1$  from [8] so that  $\varepsilon(\sigma) = \varepsilon(\sigma_1) = (-1)^m$ , where  $m = |\{i : d_{\sigma(2i-1)} = d_{\sigma(2i)} - 1\}|$ . But since  $d \in L$ ,

$$0 = \sum_j d_j = \sum_{i=1}^p (d_{\sigma(2i-1)} + d_{\sigma(2i)}) \quad (34)$$

$$= 2 \sum_{i, d_{\sigma(2i-1)} = d_{\sigma(2i)}} d_{\sigma(2i)} + 2 \sum_{i, d_{\sigma(2i-1)} = d_{\sigma(2i)} - 1} d_{\sigma(2i)} - m, \quad (35)$$

which proves that  $m$  is even. Hence  $\varepsilon(\sigma_1) = 1$ . The fact that  $\varepsilon_A = (-1)^{c(\pi)}$  comes from the analogous fact in [8].

**Remark** In the case of odd  $k = 2p + 1$ , the same discussion carries over by adding singlets to the pair partitions and with  $\sigma(k) = k$  if  $\tau(d)\sigma \in U_a$ . But equality (34) is no longer valid, which explains why the sign is not well-defined for such  $k$ .

#### 5.4.2 The cases $\tilde{B}_k$ and $\tilde{C}_k$

The argument for the cases  $\tilde{B}_k$  and  $\tilde{C}_k$  is the same; we give the details in the  $\tilde{B}_k$  case. Let us first suppose  $k$  is even,  $k = 2p$ . Suppose  $d \in L$ ,  $f$  is a sign change with support  $\bar{f}$  and  $\sigma \in \mathfrak{S}_k$  such that  $w_a = \tau(d)f\sigma \in W_a^I$ . Then,

$$w_a \{ (e_{2i-1}-e_{2i}, 0), (e_{2i}, 0), (-e_{2i-1}-e_{2i}, -1) \} = \{ (f(e_{\sigma(2i-1)}) - f(e_{\sigma(2i)}), m - n),$$

$$(f(e_{\sigma(2i)}), n), (-f(e_{\sigma(2i-1)}) - f(e_{\sigma(2i)}), -1 - m - n) \} := S,$$

with  $m = f(\sigma(2i-1))d_{\sigma(2i-1)}$  and  $n = f(\sigma(2i))d_{\sigma(2i)}$ . Thus,  $m - n \leq 0$ ,  $n \leq 0$ ,  $-1 - m - n \leq 0$ , which forces  $m = n = 0$  or  $m = -1, n = 0$ . If  $m = n = 0$ , then  $f(e_{\sigma(2i-1)}) - f(e_{\sigma(2i)}) \in \Phi^+$ ,  $f(e_{\sigma(2i)}) \in \Phi^+$ , which implies  $\sigma(2i-1), \sigma(2i) \notin \bar{f}$  and  $\sigma(2i-1) < \sigma(2i)$ . If  $m = -1, n = 0$ , then  $-f(e_{\sigma(2i-1)}) - f(e_{\sigma(2i)}) \in \Phi^+$ ,  $f(e_{\sigma(2i)}) \in \Phi^+$ , which implies  $\sigma(2i-1) \in \bar{f}$ ,  $\sigma(2i) \notin \bar{f}$  and  $\sigma(2i-1) < \sigma(2i)$ . In any case,

$$S = \{ (e_{\sigma(2i-1)} - e_{\sigma(2i)}, 0), (e_{\sigma(2i)}, 0), (-e_{\sigma(2i-1)} - e_{\sigma(2i)}, -1) \}$$

and

$$W_a^I = \left\{ \begin{array}{l} \tau(d)f\sigma \in W_a : \forall i, (d_{\sigma(2i-1)} = d_{\sigma(2i)} = 0, \sigma(2i-1), \sigma(2i) \notin \bar{f}, \\ \sigma(2i-1) < \sigma(2i)) \text{ or } (d_{\sigma(2i-1)} = 1, d_{\sigma(2i)} = 0, \sigma(2i-1) \in \bar{f}, \\ \sigma(2i) \notin \bar{f}, \sigma(2i-1) < \sigma(2i)) \end{array} \right\}.$$

Then  $\mathcal{I}$  clearly identifies with  $P_2(k)$  through the correspondence between  $\pi = \{\{i_l < j_l\}, 1 \leq l \leq p\} \in P_2(k)$  and  $A = \{(e_{i_l} - e_{j_l}, 0), (e_{j_l}, 0), (-e_{i_l} - e_{j_l}, -1); 1 \leq l \leq p\}$ . So, (C1) and (C3) are obvious by taking  $J_a = \{(e_{2i-1} - e_{2i}, 0), (-e_1 - e_2, -1)\}$ . Now,

$$U_a = \{\tau(d)f\sigma \in W_a^I : \sigma \text{ permutes pairs } (1, 2), \dots, (2p-1, 2p)\},$$

so that if  $\tau(d)f\sigma \in U_a$ ,  $\varepsilon(\tau(d)f\sigma) = \varepsilon(f)\varepsilon(\sigma) = (-1)^{|\bar{f}|}$ . But  $|\bar{f}| = \sum_i d_{\sigma(2i-1)} = \sum_j d_j$  is even, which proves (C2).

For odd  $k = 2p + 1$ ,  $\mathcal{I}$  identifies with  $P_2(k)$  through the correspondence between  $\pi = \{\{i_l < j_l\}, 1 \leq l \leq p; \{s\}\} \in P_2(k)$  and  $A = \{(e_{i_l} - e_{j_l}, 0), (e_{j_l}, 0), (-e_{i_l} - e_{j_l}, -1), 1 \leq l \leq p; (e_s, 0), (-e_s, -1)\}$ . Elements  $\tau(d)f\sigma \in U_a$  are described in the same way with the extra condition that  $\sigma(k) = k$  and  $d_k = 0, k \notin \bar{f}$  or  $d_k = 1, k \in \bar{f}$ . So the proof of (C2) carries over.

### 5.4.3 The $\tilde{D}_k$ case

Let us first suppose  $k$  is even,  $k = 2p$ . Suppose  $d \in L$ ,  $f$  is an even sign change and  $\sigma \in \mathfrak{S}_k$  such that  $w_a = \tau(d)f\sigma \in W_a^I$ . Then,

$$\begin{aligned} & w_a \{ (e_{2i-1} - e_{2i}, 0), (-e_{2i-1} + e_{2i}, -1), (e_{2i-1} + e_{2i}, 0), (-e_{2i-1} - e_{2i}, -1) \} \\ &= \{ (f(e_{\sigma(2i-1)}) - f(e_{\sigma(2i)}), m - n), (-f(e_{\sigma(2i-1)}) + f(e_{\sigma(2i)}), -1 - (m - n)), \\ & (f(e_{\sigma(2i-1)}) + f(e_{\sigma(2i)}), m + n), (-f(e_{\sigma(2i-1)}) - f(e_{\sigma(2i)}), -1 - (m + n)) \} := S, \end{aligned}$$

with  $m = f(\sigma(2i-1))d_{\sigma(2i-1)}$  and  $n = f(\sigma(2i))d_{\sigma(2i)}$ . Thus  $m - n \leq 0$ ,  $-1 - (m - n) \leq 0$ ,  $m + n \leq 0$ ,  $-1 - (m + n) \leq 0$ , which forces  $m = n = 0$  or  $m = -1, n = 0$ . If  $m = n = 0$ , then  $f(e_{\sigma(2i-1)}) \pm f(e_{\sigma(2i)}) \in \Phi^+$ , which implies  $\sigma(2i-1) \notin \bar{f}$  and  $\sigma(2i-1) < \sigma(2i)$ . If  $m = -1, n = 0$ , then  $-f(e_{\sigma(2i-1)}) \pm f(e_{\sigma(2i)}) \in \Phi^+$ , which implies  $\sigma(2i-1) \in \bar{f}$  and  $\sigma(2i-1) < \sigma(2i)$ . In any case, we have

$$\begin{aligned} S = \{ & (e_{\sigma(2i-1)} - e_{\sigma(2i)}, 0), (-e_{\sigma(2i-1)} + e_{\sigma(2i)}, -1), \\ & (e_{\sigma(2i-1)} + e_{\sigma(2i)}, 0), (-e_{\sigma(2i-1)} - e_{\sigma(2i)}, -1) \}, \end{aligned}$$

and

$$\begin{aligned} W_a^I = \{ & \tau(d)f\sigma \in W_a : \forall i, (d_{\sigma(2i-1)} = d_{\sigma(2i)} = 0, \sigma(2i-1) \notin \bar{f}, \\ & \sigma(2i-1) < \sigma(2i)) \text{ or } (d_{\sigma(2i-1)} = 1, d_{\sigma(2i)} = 0, \sigma(2i-1) \in \bar{f}, \\ & (2i-1) < \sigma(2i)) \}. \end{aligned}$$

The correspondence between  $\pi = \{ \{i_l < j_l\}, 1 \leq l \leq p \} \in P_2(k)$  and  $A = \{ (e_{i_l} - e_{j_l}, 0), (-e_{i_l} + e_{j_l}, -1), (e_{i_l} + e_{j_l}, 0), (-e_{i_l} - e_{j_l}, -1) ; 1 \leq l \leq p \}$  identifies  $\mathcal{I}$  with  $P_2(k)$ . (C1) and (C3) are obvious with  $J_a = \{ (e_{2i-1} - e_{2i}, 0), 1 \leq i \leq p; (e_{k-1} + e_k, 0) \}$ . Moreover,

$$U_a = \{ \tau(d)f\sigma \in W_a^I : \sigma \text{ permutes pairs } (1, 2), \dots, (2p-1, 2p) \},$$

which makes (C2) easy since  $\varepsilon(f) = 1$  for  $\tau(d)f\sigma \in W_a$ .

The case of odd  $k$  is an obvious modification.

### 5.4.4 The $\tilde{G}_2$ case

Call  $\alpha_1 = e_1 - e_2$ ,  $\alpha_2 = 2e_3 - e_1 - e_2 = \tilde{\alpha}$  and take  $J_a = \{ (\alpha_1, 0), (-\alpha_2, -1) \}$ . We remark that  $I$  can be written

$$\{ (\alpha_1, 0), (-\alpha_1, -1), (\alpha_2, 0), (-\alpha_2, -1) \} \text{ with } \alpha_1 \text{ short, } \alpha_2 \text{ long, } \alpha_1 \perp \alpha_2. \quad (36)$$

If  $w_a = \tau(d)w \in W_a^I$  then  $(w\alpha_i, d) \in \mathbb{Z}$ ,  $(w\alpha_i, d) \leq 0$  and  $-1 - (w\alpha_i, d) \leq 0$ , which imposes  $(w\alpha_i, d) \in \{0, -1\}$  for  $i = 1, 2$ . Thus,  $A = w_a I$  can also be written as in (36) for some  $\alpha'_1, \alpha'_2$ . This guarantees condition (C3) and if



$J_a \subset A$  then obviously  $\alpha_1 = \alpha'_1$ ,  $\alpha_2 = \alpha'_2$  so that  $A = I$ , which proves condition (C1). Writing  $I$  as in (36) allows us to see that if  $w_a = \tau(d)w \in W_a$ , then  $w_a I = \{(w\alpha_1, m_1), (-w\alpha_1, -1 - m_1), (w\alpha_2, m_2), (-w\alpha_2, -1 - m_2)\}$  where  $m_i = (w\alpha_i, d) \in \mathbb{Z}$ . Since  $W$  sends long (short) roots to long (short) roots,  $w_a \in U_a$  implies  $w\alpha_i \in \{\pm\alpha_i\}$  for  $i = 1, 2$ . If  $w\alpha_i = \alpha_i$  for  $i = 1, 2$  (respectively  $w\alpha_i = -\alpha_i$  for  $i = 1, 2$ ), then  $w = \text{id}$  (respectively  $w = -\text{id}$ ) and  $\varepsilon(w) = 1$  (recall that  $\dim V = 2$ ). If  $w\alpha_1 = \alpha_1$  and  $w\alpha_2 = -\alpha_2$  then  $(\alpha_1, d) = 0$  and  $(\alpha_2, d) = 1$ . This implies  $d = (-1/6, -1/6, 1/3) \notin L$ , which is absurd. The same absurdity occurs if  $w\alpha_1 = -\alpha_1$  and  $w\alpha_2 = \alpha_2$ .

For the determination of  $\mathcal{I}$ , it is easy to see that the sets of the form (36) are  $I, A_1, A_2$ . The sign of the transformation sending  $(\alpha_1, \alpha_2)$  to  $(e_3 - e_1, -2e_2 + e_1 + e_3)$  is 1 so that  $\varepsilon_{A_1} = -1$  and  $A_2$  is obtained from  $A_1$  by transposing  $e_1$  and  $e_2$ , which finishes the proof.

#### 5.4.5 The $\tilde{F}_4$ case

Call  $\alpha_1 = e_2 - e_3$ ,  $\alpha'_1 = e_3$ ,  $\alpha_2 = e_1 - e_4$ ,  $\alpha'_2 = e_4$ . Then  $I$  can be written

$$\{(\alpha_1, 0), (-\alpha_1, -1), (\alpha'_1, 0), (\alpha_2, 0), (-\alpha_2, -1), (\alpha'_2, 0)\}, \quad (37)$$

with  $\alpha_1, \alpha_2$  long,  $\alpha'_1, \alpha'_2$  short,  $\{\alpha_1, \alpha'_1\} \perp \{\alpha_2, \alpha'_2\}$  and  $(\alpha_i, \alpha'_i) = -1$ . The same kind of reasoning as in the  $\tilde{G}_2$  case shows conditions (C1) and (C3), with  $J_a = \{\alpha_1, \alpha'_2\}$ . Let us prove (C2). If  $w_a = \tau(d)w \in U_a$ , then  $w_a I =$

$$\{(w\alpha_1, m_1), (-w\alpha_1, -1 - m_1), (w\alpha'_1, m'_1), (w\alpha_2, m_2), (-w\alpha_2, -1 - m_2), (w\alpha'_2, m'_2)\},$$

with  $m_i = (w\alpha_i, d)$ ,  $m'_i = (w\alpha'_i, d)$ . Since  $w$  sends long (short) roots to long (short) roots, necessarily  $w\{\alpha'_1, \alpha'_2\} = \{\alpha'_1, \alpha'_2\}$  and  $m'_1 = m'_2 = 0$ .

Suppose  $w\alpha'_i = \alpha'_i$ ,  $i = 1, 2$ . Since  $(w\alpha_2, \alpha'_1) = (\alpha_2, \alpha'_1) = 0 \neq -1$ , we have  $w\alpha_1 \in \{\alpha_1, -\alpha_1\}$  and  $w\alpha_2 \in \{\alpha_2, -\alpha_2\}$ . If  $w\alpha_1 = -\alpha_1$ ,  $w\alpha_2 = \alpha_2$  then  $m_1 = 1$ ,  $m_2 = 0 = m'_1 = m'_2$ , which leads to  $d = (0, 1, 0, 0) \notin L$ , absurd! If  $w\alpha_1 = \alpha_1$ ,  $w\alpha_2 = -\alpha_2$ , a similar reasoning leads to the absurdity  $d = (1, 0, 0, 0) \notin L$ . Hence,  $w\alpha_1 = \alpha_1$ ,  $w\alpha_2 = \alpha_2$  or  $w\alpha_1 = -\alpha_1$ ,  $w\alpha_2 = -\alpha_2$ . Then, using the basis  $(\alpha_1, \alpha'_1, \alpha_2, \alpha'_2)$ ,  $\varepsilon(w) = 1$  is easily checked.

Suppose now  $w\alpha'_1 = \alpha'_2$ ,  $w\alpha'_2 = \alpha'_1$ . Similar arguments show that  $w\alpha_2 \in \{\alpha_1, -\alpha_1\}$  and  $w\alpha_1 \in \{\alpha_2, -\alpha_2\}$ . If  $w\alpha_1 = \alpha_2$ ,  $w\alpha_2 = \alpha_1$  or  $w\alpha_1 = -\alpha_2$ ,  $w\alpha_2 = -\alpha_1$  then  $\varepsilon(w) = 1$ . Suppose  $w\alpha_1 = \alpha_2$ ,  $w\alpha_2 = -\alpha_1$ , then  $m_1 = 0$ ,  $m_2 = -1$ , which, as before, leads to  $d = (0, 1, 0, 0) \notin L$ . If  $w\alpha_1 = -\alpha_2$ ,  $w\alpha_2 = \alpha_1$ , then  $m_1 = -1$ ,  $m_2 = 0$ , which also gives  $d = (1, 0, 0, 0) \notin L$ .  $\square$

## 5.5 Proposition 1

The definition of the Pfaffian is given in the appendix. We refer to (17) for even  $k$  and (8) and for odd  $k$ .  $\square$

## 5.6 Proposition 11

We will use the following expansions involving the exit time  $T_{(0,1)}$  from  $(0, 1)$  and the hitting times  $T_0$  and  $T_1$  of 0 and 1 respectively for one-dimensional Brownian motion: for  $(x, t) \in (0, 1) \times [0, \infty)$ ,

$$\begin{aligned}\phi(x, t) &:= \mathbb{P}_x(T_{0,1} > t) = \sum_{l \in 2\mathbb{N}+1} c_l e^{-\lambda_l t} \sin(\pi l x) \\ \psi(x, t) &:= \mathbb{P}_x(T_{0,1} > t) + 2\mathbb{P}_x(T_0 > T_1) - 2\mathbb{P}_x(T_0 > T_1 > t) \\ &= \sum_{l \in 2\mathbb{N}} c_l e^{-\lambda_l t} \sin(\pi l x)\end{aligned}\tag{38}$$

with  $c_l = 4/(l\pi)$ ,  $\lambda_l = (l\pi)^2/2$  and the definition  $c_l \sin(\pi l x) = 2x$  when  $l = 0$ . The first expansion may be found in, for example, [3]; in the case of even  $k$ , it may be used to rewrite (17) into the form (39). The second expansion is obtained using

**Lemma 26.** *If  $X$  is Brownian motion and  $\beta = e_i - e_j$  then*

$$\mathbb{P}_x[T_\beta > T_{(\beta,1)} > t] = 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\pi n} e^{-\pi^2 n^2 t} \sin(\pi n x_{ij})$$

where  $x_{ij} = x_i - x_j \in (0, 1)$ .

**Proof** The series satisfies the diffusion equation for  $(x_{ij}, t) \in (0, 1) \times (0, \infty)$ , takes the value 0 if  $x_{ij} \in \{0, 1\}$ , and equals  $x_{ij}$  if  $t = 0$ .  $X_{ij} := X_i - X_j$  is a Brownian motion with the same diffusion coefficient. Therefore by applying for example Theorem 4.14 of [3], the series equals

$$\begin{aligned}\mathbb{E}_x[X_{ij}(t); T_\beta \wedge T_{(\beta,1)} > t] &= \mathbb{E}_x[\mathbb{P}_x[T_\beta > T_{(\beta,1)} > t | X(t), \mathbf{1}_{T_\beta \wedge T_{(\beta,1)} > t}]] \\ &= \mathbb{P}_x[T_\beta > T_{(\beta,1)} > t].\end{aligned}$$

$\square$

We record the following corollary, which follows from integration, interchanging integration with summation, and inversion of Fourier series:

**Corollary 27.** *Under the conditions of Lemma 26,*

$$\int_0^\infty \mathbb{P}_x[T_\beta > T_{(\beta,1)} > t] dt = \frac{1}{6} x_{ij} (1 - x_{ij}^2).$$

In the case of odd  $k$ , the second expansion in (38) may be inserted in Proposition 10 to give (39):

$$\begin{aligned}\mathbb{P}_x(\tilde{T} > t) &= \sum_{\pi=\{\{i_s < j_s\}, 1 \leq s \leq m\}} (-1)^{c(\pi)} \prod_{s=1}^m \left( \sum_{l \in \mathbb{O}} c_l e^{-2\lambda_l t} \sin(\pi l x_{i_s j_s}) \right) \quad (39) \\ &= \sum_{\pi=\{\{i_s < j_s\}, 1 \leq s \leq m\}} (-1)^{c(\pi)} \sum_{l \in \mathbb{O}^m} e^{-\pi^2(l_1^2 + \dots + l_m^2)t} \prod_{s=1}^m c_{l_s} \sin(\pi l_s x_{i_s j_s})\end{aligned}$$

for  $x \in \mathcal{A}$ , where  $m = \lfloor k/2 \rfloor \in \mathbb{N}$ ,  $x_{ij} = x_i - x_j$ ,  $\mathbb{O} = 2\mathbb{N} + 1$  if  $k$  is even and  $\mathbb{O} = 2\mathbb{N}$  if  $k$  is odd. Now for  $\pi = \{\{i_s < j_s\}, 1 \leq s \leq m\}$  define

$$G_r(x, \pi) = \sum_{l \in \mathbb{O}^m, N(l)=r} \prod_{s=1}^m c_{l_s} \sin(\pi l_s x_{i_s j_s}) \quad (40)$$

where  $N(l) = l_1^2 + \dots + l_m^2$ , and let  $F_r(x) = \sum_{\pi \in P_2(k)} (-1)^{c(\pi)} G_r(x, \pi)$ . (Since the sum defining  $G_r(x, \pi)$  runs over a  $\mathfrak{S}_m$ -invariant set of indices, it does not depend on the enumeration of the blocks of  $\pi$  but only on  $\pi$  itself.) With those definitions we can write

$$\mathbb{P}_x(\tilde{T} > t) = \sum_{r>0} e^{-\pi^2 r t} F_r(x) \quad (41)$$

(note that by Proposition 2.4 of [8],  $\sum_{\pi \in P_2(k)} (-1)^{c(\pi)} \prod_{s=1}^m x_{i_s j_s} = 0$  and so the terms corresponding to  $r = 0$  cancel.) As for expectations, we have

$$\mathbb{E}_x(\tilde{T}) = \int_0^\infty \mathbb{P}_x(\tilde{T} > t) dt = \sum_{r>0} \frac{1}{r\pi^2} F_r(x)$$

and the result follows.  $\square$

When  $k = 2$  the previous formula becomes

$$\mathbb{E}_x(\tilde{T}) = \sum_{n \in \mathbb{N}} \frac{4}{\pi^3} \frac{\sin(\pi(2n+1)x_{12})}{(2n+1)^3} = \frac{1}{2} x_{12} (1 - x_{12}), \quad (42)$$

$0 < x_{12} < 1$ , which is a well-known formula in Fourier series. When  $k = 3$  we may use the above and Corollary 27 to obtain

$$\mathbb{E}_x(\tilde{T}) = \sum_{\pi=\{\{i_s < j_s\}\}} (-1)^{c(\pi)} \sum_{n \in \mathbb{N}} \frac{4}{\pi^3} \frac{\sin(2\pi n x_{ij})}{(2n)^3} = x_{12} x_{23} (1 - x_{13}), \quad (43)$$

$0 < x_{ij} < 1$ . It is easy to check that (42) and (43) both solve Poisson's equation  $\frac{1}{2}\Delta u = -1$  inside the interval and an equilateral triangle respectively and vanish on the boundary, which confirms that they are the expected exit times for Brownian motion from these domains. Formula (43) has also been obtained using scaling limits for random walks (see [1, 5]).

## 5.7 The reflection principle and De Bruijn Formulae

### 5.7.1 Proposition 14

From (14), if  $T_A$  is the exit time of Brownian motion from  $F_A$  then

$$\mathbb{P}_x[T_A > t] = \int_{F_A} \sum_{\omega \in W_A} \varepsilon(\omega) p_t(x, \omega y) dy \quad (44)$$

where  $p_t$  is the Brownian transition density and  $x \in F_A$ . The finite case was proved in [8]; in the affine case it is easy to check that the same proof applies.

### 5.7.2 Propositions 15 and 16

We treat first the case of odd  $k$ . Let  $\beta \in \Phi^+ = \{e_i - e_j : 1 \leq i < j \leq k\}$  and  $x \in \mathcal{A}$ . Then  $(x, \beta) \in (0, 1)$  and from (44), for  $k \geq 1$

$$\begin{aligned} \mathbb{P}_x[T_{(\beta, k)} > t] &= \int_{\langle y, \beta \rangle < k} p_t(x, y) - p_t(x, s_\beta y + k\beta) dy \\ &= \int_{\langle u, \beta \rangle > -k} p_t(x, s_\beta u) - p_t(x, u + k\beta) du \end{aligned}$$

where  $u = s_\beta y$ . Also if  $k \leq 0$  then

$$\mathbb{P}_x[T_{(\beta, k)} > t] = \int_{\langle y, \beta \rangle > k} p_t(x, y) - p_t(x, s_\beta y + k\beta) dy. \quad (45)$$

Write  $\beta = e_i - e_j$ . Rewriting Theorem 8 using Lemma 24, equation (45) and using the identification of  $\mathcal{I}$  with  $P_2(k)$  from section 4.1 we have

$$\begin{aligned} \mathbb{P}_x[\tilde{T} > t] &= \sum_{\pi \in P_2(k)} (-1)^{c(\pi)} \prod_{\{i < j\} \in \pi} \left( \int_{y_i > y_j} p_{ij}(0) dy_i dy_j \right. \\ &\quad \left. + \sum_{k=1}^{\infty} \int_{y_i - y_j > -k} p_{ij}(0) + p_{ij}(k) dy_i dy_j \right) \end{aligned} \quad (46)$$

where  $p_{ij}(k) = \psi(x_i, y_i + k)\psi(x_j, y_j - k) - \psi(x_i, y_j - k)\psi(x_j, y_i + k)$  and  $\psi(x, y) = \frac{1}{\sqrt{2\pi t}} e^{-(x-y)^2/2t}$ . Now  $\int_{-k < y_i - y_j < k} p_{ij}(0) dy_i dy_j = 0$  and making the substitution  $(u_i, u_j) = (y_i + k, y_j - k)$  we have

$$\int_{y_i - y_j > -k} p_{ij}(k) dy_i dy_j = \int_{u_i - u_j > k} p_{ij}(0) du_i du_j,$$

so the infinite sum in (46) may be written  $2 \sum_{k=1}^{\infty} \int_{y_i - y_j > k} p_{ij}(0) dy_i dy_j$ . From (44) we have the alternative expression

$$\mathbb{P}_x[\tilde{T} > t] = \int_{\mathcal{A}} \sum_{\omega \in W_a} \varepsilon(\omega) p_t(x, \omega y) dy$$

so integrating both expressions over  $\mathbb{R}^k$  with respect to  $f_i(x_i)dx_i$ ,  $i = 1, \dots, k$  and applying Fubini's theorem,

$$\int_{\mathcal{A}} \sum_{\omega \in W_a} \varepsilon(\omega) P_t f(\omega y) dy = \sum_{\pi \in P_2(k)} (-1)^{c(\pi)} \int_{\mathbb{R}} f_{l_\pi} \prod_{\{i < j\} \in \pi} \left( \int_{y_i > y_j} P_{ij} dy_i dy_j + 2 \sum_{k=1}^{\infty} \int_{y_i - y_j > k} P_{ij} dy_i dy_j \right)$$

where  $\{l_\pi\}$  is the singlet in the partition  $\pi$  and  $P_{ij} = P_t f_i(y_i) P_t f_j(y_j) - P_t f_i(y_j) P_t f_j(y_i)$ . To complete the proof for the case of odd  $k$  we obtain uniform bounds in  $t$  to justify the use of dominated convergence to let  $t \rightarrow 0$  inside the infinite sum, and finally apply the definition of the Pfaffian. Dividing the domain of integration into  $(-k + \sqrt{t}, k - \sqrt{t})$  and its complement and applying the bound  $\int p_{ij}(0) dy_i dy_j \leq 2$  on the latter we have for  $t < 1/4$

$$\begin{aligned} \left| \int_{y_i - y_j > k} P_{ij} dy_i dy_j \right| &\leq \int_{x_i, x_j \in \mathbb{R}} \int_{y_i - y_j > k} |p_{ij}(0) f_i(x_i) f_j(x_j)| dy_i dy_j dx_i dx_j \\ &\leq \int_{x_i - x_j < k - \sqrt{t}} \int_{y_i - y_j > k} \psi(x_i, y_i) \psi(x_j, y_j) (|f_i(x_i) f_j(x_j)| + |f_i(x_j) f_j(x_i)|) \\ &\quad dy_i dy_j dx_i dx_j + 2 \int_{x_i - x_j \in (-\infty, -k + 1/2) \cup (k - 1/2, \infty)} |f_i(x_i) f_j(x_j)| dx_i dx_j \end{aligned}$$

and  $\int_{x_i - x_j \in (-\infty, -k) \cup (k, \infty)} |f_i(x_i) f_j(x_j)| dx$  is summable in  $k$  by assumption. The standard estimate for the tail of the Gaussian distribution gives

$$\int_{y_i - y_j > k} \psi(x_i, y_i) \psi(x_j, y_j) dy_i dy_j \leq e^{-(k - (x_i - x_j))^2}$$

when  $x_i - x_j < k - \sqrt{t}$ , and  $\int_{x \in \mathbb{R}} e^{-(k - (x_i - x_j))^2} |f(x)| dx$  is summable in  $k$ .

When  $k$  is even we have a consistent subset  $I$  as described in section 4.1 and so Proposition 14 applies. The proof is similar to that in section 7.6.1 of [8], with the difference that here we have the bijection

$$(l \in L_\pi, \eta \in \{\pm 1\}^\pi) \mapsto w_{l, \eta} = \tau(l) \prod_{\{i < j\} \in \pi} \tau'_{ij} \in W_A$$

where  $\pi \in P_2(k)$  is the pair partition associated with  $A \in \mathcal{I}$ , and  $L_\pi$  is the coroot lattice associated with the affine Weyl group  $W_A$ ; and now  $F_A$  corresponds with  $F_\pi = \cap_{\{i < j\} \in \pi} \{y : 0 < y_i - y_j < 1\}$ .  $\square$

## 6 Eigenfunctions and eigenvalues for alcoves

It follows from equation (41) that  $F_r$  is a real eigenfunction for the Dirichlet Laplacian on the alcove of type  $\tilde{A}_{k-1}$ , with eigenvalue  $-2\pi^2 r$ . As an

example, when  $k = 3$  the alcove is the equilateral triangle and we have

$$\frac{1}{c_{2n}} F_r(x) = \begin{cases} \sin(2\pi n x_{12}) + \sin(2\pi n x_{23}) - \sin(2\pi n x_{13}) & \text{if } r = 4n^2 \\ 0 & \text{otherwise,} \end{cases}$$

giving the eigenfunctions with simple eigenvalues (see [14]), a feature which can be anticipated from the symmetry of the equilateral triangle. Bérard [4] obtained a general formula for the eigenfunctions of the Dirichlet and Neumann Laplacians for alcoves of any type, and we provide here a characterisation of the real eigenfunctions.

Defining

$$f_p(x) = \sum_{w \in W} \varepsilon(w) \exp(2\pi i \langle x, wp \rangle), \quad g_p(x) = \sum_{w \in W} \exp(2\pi i \langle x, wp \rangle), \quad (47)$$

the eigenfunctions for the Dirichlet Laplacian on  $\mathcal{A}$  are  $\{f_p : p \in \mathcal{P} \cap \mathcal{C}\}$ , where  $\varepsilon(w) = \det w$  and  $\mathcal{P} = \{x \in V : \langle \alpha^\vee, x \rangle \in \mathbb{Z} \ \forall \ \alpha \in \Phi\}$ , and the eigenfunctions for the Neumann Laplacian on  $\mathcal{A}$  are  $\{g_p : p \in \mathcal{P} \cap \overline{\mathcal{C}}\}$ .

**Remark** It is immediate from (47) that if  $g_p$  is real then for every  $y \in \mathcal{A}$  we have  $g_p(y) < \sup_{x \in \partial \mathcal{A}} g_p(x)$ . The ‘Hot Spots’ conjecture of J. Rauch (see [2]) is therefore true for alcoves. Note that in the two-dimensional case, the alcoves are the equilateral triangle and the right triangles with an angle of either  $\pi/4$  or  $\pi/3$ .

**Proposition 28.** (i) For  $p \in \mathcal{P} \cap \mathcal{C}$ , the eigenfunction  $f_p$  of the Dirichlet Laplacian on  $\mathcal{A}$  is real iff

$$\exists w_1 \in W \text{ such that } w_1 p = -p. \quad (48)$$

If (48) holds then, up to a constant factor,

$$f_p(x) = \sum_{w \in W} \varepsilon(w) cs(2\pi \langle x, wp \rangle)$$

where  $cs = \sin$  if  $\varepsilon(w_1) = -1$  and  $cs = \cos$  if  $\varepsilon(w_1) = 1$ .

(ii) For  $p \in \mathcal{P} \cap \overline{\mathcal{C}}$ , the eigenfunction  $g_p$  of the Neumann Laplacian on  $\mathcal{A}$  is real iff (48) holds and then, up to a constant factor,

$$g_p(x) = \sum_{w \in W} \cos 2\pi \langle x, wp \rangle.$$

*Proof.* (i) We have

$$f_p(x) = \sum_{w \in W} \varepsilon(w) \cos 2\pi \langle x, wp \rangle + i \sum_{w \in W} \varepsilon(w) \sin 2\pi \langle x, wp \rangle.$$

Suppose first that  $w_1 p = -p$  for some  $w_1 \in W$ . Then by conjugation, for any  $w \in W$  there exists  $v_w \in W$  such that  $v_w(wp) = -wp$ . The orbit  $Wp$  may therefore be partitioned into pairs  $\{wp, -wp\}$ , and

$$cs(2\pi \langle x, wp \rangle) \pm cs(2\pi \langle x, -wp \rangle) = 0$$

where  $\pm = +, -$  if  $\text{cs} = \sin, \cos$  respectively. The sufficiency of condition (48) is proved by noting that  $\forall w \in W, \varepsilon(v_w w) = \varepsilon(v_w)\varepsilon(w) = \varepsilon(w_1)\varepsilon(w)$ .

Conversely, suppose that

$$\sum_{w \in W} \varepsilon(w) \text{cs} 2\pi \langle x, wp \rangle = 0 \quad \forall x \in V. \quad (49)$$

By restricting  $x$  to a ray  $x = tr$  ( $t \in \mathbb{R}$ ) chosen such that  $\langle r, wp \rangle = \langle r, p \rangle$  only when  $w = \text{Id}$  and  $\langle r, wp \rangle = -\langle r, p \rangle$  only when  $wp = -p$  and appealing to linear independence, we conclude that (48) holds. Part (ii) is proved similarly.  $\square$

Using standard facts about the longest element of a Weyl group (see [13]) we obtain

**Corollary 29.** *For the cases  $W = A_1, B_k, C_k, D_{2k}, E_7, E_8, F_4, G_2, H_3$  and  $H_4$ , all the eigenfunctions of the Laplacian on  $\mathcal{A}$  with Dirichlet or Neumann boundary conditions are real. In all other cases, the eigenfunctions  $f_p, g_p$  given by (47) are real iff  $p = \tau(p)$ , where  $\tau$  is the unique involution of the Coxeter graph of  $W$ .*

The root systems covered by the second case of Corollary 29 are

**Type  $A_{k-1}$ ,  $k > 2$**  Here  $\tau(e_i - e_{i+1}) = e_{k-i} - e_{k-i+1}$  and so we require  $p = \sum_{i=1}^{k-1} a_i(e_i - e_{i+1})$  with  $a_i = a_{k-i} \forall 1 \leq i \leq k-1$ .

**Type  $D_{2k+1}$**  Here  $\tau$  leaves  $e_i - e_{i+1}$  invariant for  $1 \leq i \leq 2k-1$ , and  $\tau(e_{2k} - e_{2k+1}) = e_{2k} + e_{2k+1}$ . We therefore require  $p = \sum_{i=1}^{2k} a_i(e_i - e_{i+1}) + a_{2k+1}(e_{2k} + e_{2k+1})$  with  $a_{2k} = a_{2k+1}$ .

**Proof of Proposition 12** Defining  $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$ , we have  $\rho \in \mathcal{P} \cap \mathcal{C}$  (see for example [13]). Then setting  $p = \rho$  in (47), the Weyl identity gives that up to a constant factor,

$$f_\rho(x) = \prod_{\alpha \in \Phi^+} \sin(\pi \langle x, \alpha \rangle). \quad (50)$$

The next lemma establishes the final claim of Proposition 12.

**Lemma 30.** *Suppose that  $F(X) = F(X_j)_{j \in J}$  is a polynomial in the  $(\sin X_j, \cos X_j)_{j \in J}$  which vanishes whenever  $\sin X_j$  vanishes. Then  $\sin X_j$  divides  $F(X)$  in the ring of trigonometric polynomials.*

*Proof.* Let  $F(X) = P(e^{iX_j}, e^{-iX_j})_{j \in J} \in R := \mathbb{C}[e^{iX_j}, e^{-iX_j}; j \in J]$ . The given cancellation property assures that  $P$  is divisible in  $R$  by the monic polynomial  $e^{iX_j} - 1$  and that the quotient is divisible by  $e^{iX_j} + 1$ . Hence  $P$  is divisible in  $R$  by  $\frac{e^{2iX_j} - 1}{2ie^{iX_j}} = \sin X_j$ .  $\square$

Since the eigenfunctions are alternating under the action of the affine Weyl group (see for example [4]), and putting  $J = \Phi^+$  and  $X_\alpha = \pi \langle \alpha, x \rangle$ , the Lemma applies. Using continuity and Lemma 30 again establishes the final claim of Proposition 12.  $\square$

**Remark** In the type  $\tilde{A}$  case, the principal eigenfunction was obtained by Hobson and Werner in [11]; we give a direct proof in the appendix. See also [7].

## 7 Appendix

### 7.1 Direct proof of Proposition 12 in the type $\tilde{A}$ case

Set  $x_{ij} = x_i - x_j$  and  $h(x) = \prod_{1 \leq i < j \leq k} \sin x_{ij}$ . Computation of the logarithmic derivative gives

$$\partial_i h = h \sum_{j \neq i} \frac{\cos x_{ij}}{\sin x_{ij}},$$

which yields

$$\begin{aligned} \partial_i^2 h &= h \left\{ \sum_{j, l \neq i} \frac{\cos x_{ij} \cos x_{il}}{\sin x_{ij} \sin x_{il}} + \sum_{j \neq i} \left( -1 - \frac{\cos^2 x_{ij}}{\sin^2 x_{ij}} \right) \right\} \\ &= h \left\{ \sum_{j \neq l \neq i} \frac{\cos x_{ij} \cos x_{il}}{\sin x_{ij} \sin x_{il}} - (k-1) \right\}, \end{aligned}$$

so that  $\Delta h = h(S(x) - k(k-1))$  with

$$S(x) = \sum' \frac{\cos x_{ij} \cos x_{il}}{\sin x_{ij} \sin x_{il}},$$

where  $\sum'$  runs over  $i, j, l$  pairwise distinct. By circular permutation, we get

$$\begin{aligned} 3S(x) &= \sum' \frac{\cos x_{ij} \cos x_{il}}{\sin x_{ij} \sin x_{il}} + \frac{\cos x_{jl} \cos x_{ji}}{\sin x_{jl} \sin x_{ji}} + \frac{\cos x_{li} \cos x_{lj}}{\sin x_{li} \sin x_{lj}} \\ &= \sum' \frac{\cos x_{ij} \cos x_{il} \sin x_{jl} - \cos x_{jl} \cos x_{ij} \sin x_{il} + \sin x_{ij} \cos x_{il} \cos x_{jl}}{\sin x_{ij} \sin x_{il} \sin x_{jl}}. \end{aligned}$$

But trigonometry shows that each term in the previous sum equals  $-1$ , so that  $S(x) = -k(k-1)(k-2)/3$ , which concludes the proof.

### 7.2 The Pfaffian

For completeness we define the Pfaffian. If  $\text{car } \mathbb{K} \neq 2$ , any skew-symmetric matrix  $A \in \mathcal{M}_n(\mathbb{K})$  can be written  $A = PDP^t$  with  $P \in GL(n, \mathbb{K})$ ,  $D = \text{diag}(B_1, \dots, B_q)$  and  $B_l = 0 \in \mathbb{K}$  or  $B_l = J = (j-i)_{1 \leq i, j \leq 2} \in \mathcal{M}_2(\mathbb{K})$ . Hence, if  $n$  is odd,  $\det A = 0$ . If  $n$  is even, one can use the previous decomposition to prove



**Proposition 31.** *There exists a unique polynomial  $\text{Pf} \in \mathbb{Z}[X_{ij}, 1 \leq i < j \leq n]$  such that if  $A = (a_{ij})$  is a skew-symmetric matrix of size  $n$ ,  $\det A = \text{Pf}(A)^2$  and  $\text{Pf}(\text{diag}(J, \dots, J)) = 1$ .*

The Pfaffian has an explicit expansion in terms of the matrix coefficients:

**Proposition 32.**

$$\text{Pf}(A) = \sum_{\pi \in P_2(n)} (-1)^{c(\pi)} \prod_{\{i < j\} \in \pi} a_{ij} = \frac{1}{2^n (n/2)!} \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) \prod_{i=1}^{n-1} a_{\sigma(i)\sigma(i+1)}.$$

For more on Pfaffians and their properties, see [9, 15].

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